BAROCLINIC INSTABILITY
IN DIFFERENTIALLY ROTATING STARS

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Origin of the baroclinic instability
Eigenvalue equations
Excitation of $r$- and $g$- modes
The effect of composition gradient
Helicity and possibility of dynamos
Is it observable?
Hydrodynamical equilibrium in rotating star

Equilibrium condition:

\[ \frac{1}{\rho} \nabla P = g^* \]

\[ g^* = g + r \sin \theta \Omega \text{e}_\phi \times \Omega \]

\[ \sin \theta \frac{\partial \Omega^2}{\partial z} = -\frac{1}{\rho^2} (\nabla \rho \times \nabla P)_\phi = \frac{1}{c_p} (\nabla s \times g^*)_\phi \]

Barotropic stratification

Sufficiently large differential rotation, \( \Delta \Omega \sim 0.1\Omega \), can be unstable (Watson 1981; Dziembowski & Kosovichev 1987; Charbonneau et al. 1999; ...)

Baroclinic stratification

The “stable” stratification - if baroclinic - can provoke an instability (Shibahashi 1980; Tassoul & Tassoul 1983; ...)

\[ V = e_\phi r \sin \theta \Omega \]
Basic assumptions/approximations

- Shellular rotation
\[ \Omega(r), \quad q = -\frac{r \, d\Omega}{\Omega \, dr} \]

- Disturbances are global in horizontal dimensions but short-scaled in radius
\[ \mathbf{u}, \mathcal{S} \sim \exp(-i\omega t + im\phi + ikr) \]
\[ \mu = \cos \theta \]

- Non-compressive disturbances but entropy perturbations due to radial displacements are allowed (Boussinesq approximation)

- Scalar potentials are used to specify toroidal (W) and poloidal (V) parts of the flow:
\[ \mathbf{u} = \frac{e_\phi}{r^2} (\hat{L} W) - \frac{e_\theta}{r} \left( \frac{1}{\sin \theta} \frac{\partial W}{\partial \phi} + \frac{\partial^2 V}{\partial r \partial \phi} \right) + \frac{e_\phi}{r} \left( \frac{\partial W}{\partial \theta} - \frac{1}{\sin \theta} \frac{\partial^2 V}{\partial r \partial \phi} \right) \]
\[ \hat{L} = \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial}{\partial \mu} + \frac{1}{1 - \mu^2} \frac{\partial^2}{\partial \phi^2} \]

Equations of linear stability (eigenvalue) problem

- Toroidal flow
\[ \hat{\omega}(\hat{L} W) = -i\frac{\epsilon_\nu}{\lambda^2} (\hat{L} W) + 2mW - 2\mu(\hat{L} V) - 2(1 - \mu^2) \frac{\partial V}{\partial \mu} \]

- Poloidal flow
\[ \hat{\omega}(\hat{L} V) = -i\frac{\epsilon_\nu}{\lambda^2} (\hat{L} V) - \lambda^2 (\hat{L} S) + 2mV - 2\mu(\hat{L} W) - 2(1 - \mu^2) \frac{\partial W}{\partial \mu} \]

- Entropy
\[ \hat{\omega} S = -i\frac{\epsilon_\nu}{\lambda^2} S + \hat{L} V + \frac{Q}{\lambda} \left( mW - (1 - \mu^2) \frac{\partial V}{\partial \mu} \right) \]

Two governing parameters:
\[ \lambda = \frac{N}{\Omega kr}, \quad Q = 2q \frac{\Omega}{N} \]

Normalized diffusivities: \[ \epsilon_\chi = 10^{-4}, \quad \epsilon_\nu = 2 \times 10^{-10} \]
Symmetry properties

Equator-symmetric Sm-modes

\[ V(\mu) = V(-\mu) \]
\[ W(\mu) = -W(-\mu) \]
\[ S(\mu) = S(-\mu) \]

Equator-antisymmetric Am-modes

\[ V(\mu) = -V(-\mu) \]
\[ W(\mu) = W(-\mu) \]
\[ S(\mu) = -S(-\mu) \]

Eigenvalue equations are symmetric under the transformation:

\[ (q, m, \omega, W, V, S) \rightarrow (-q, -m, -\omega^*, -W^*, V^*, -S^*) \]

Unstable modes are expected to possess kinetic helicity

\[ H_{rel} = \langle u \cdot (\nabla \times u) \rangle / (k \bar{u}^2) \]

\[ \langle X \rangle = \frac{1}{2\pi} \int X d\phi, \quad \bar{u}^2 = \frac{1}{2} \int \langle u^2 \rangle d\mu \]
Two modes of stable oscillations
(uniform rotation, zero diffusion, $N >> \Omega$)

**Toroidal $r$-modes**

\[ \omega^r_{lm} = -\frac{2m\Omega}{l(l+1)} \]

**Poloidal $g$-modes**

\[ \omega^g_{lm} = \pm \frac{N}{kr} \sqrt{l(l+1)} \]
Stability map

\[ \frac{S u_r}{\sqrt{u_r^2 S^2}} > 0 \] for all unstable modes
Baroclinic instability as a stability loss to excitation of $r$- and $g$-modes

Parameters of unstable disturbances for $\lambda = 3$ and $Q = 10^{-3}$.

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\gamma$, $10^{-4}$</th>
<th>$R(\tilde{\omega})$</th>
<th>$\tilde{\omega}^r$</th>
<th>$\tilde{\omega}^g$</th>
<th>$\overline{u_p^2/u_t^2}$</th>
<th>$\overline{S u_r}/\sqrt{u_t^2} S^2$</th>
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<tbody>
<tr>
<td>A0</td>
<td>8.41</td>
<td>4.34</td>
<td>4.24</td>
<td>24.0</td>
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<td>A3</td>
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<td>A10</td>
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<td>A-1</td>
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<td>1</td>
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<tr>
<td>A-3</td>
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<td>0.199</td>
<td>0.2*</td>
<td>$9.62 \times 10^{-7}$</td>
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<tr>
<td>A-10</td>
<td>0.507</td>
<td>0.0952</td>
<td>0.0952*</td>
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<td>0.155</td>
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<tr>
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Kinetic energy of disturbances is the sum of energies of their toroidal and poloidal parts:

$$\overline{u^2} = \overline{u_p^2} + \overline{u_t^2} = \frac{1}{4} \sum_l l(l + 1) (|V_l|^2 + |W_l|^2)$$
Growth rates

\[ \frac{N}{(\Omega r k)} = 3 \]

\[ 2q \Omega / N = 0.001 \]
Dependence on the thermal diffusivity

\[ \frac{N}{(\Omega r k)} = 3 \]

The lines are marked by the values of (normalized) thermal conductivity

Composition stratification changes the effective buoyancy frequency:

\[ N^2 \rightarrow N_{\text{eff}}^2 = N^2 + N_{\mu}^2 , \]

\[ N_{\mu}^2 = -\frac{g \, d\mu}{\mu \, dr} \]
Eigenmodes

\( \hat{\lambda} = 3, \quad Q = 0.001 \)

**Toroidal \( r \)-mode A-1**

\[
\frac{\bar{u}_D^2}{\bar{u}_T^2} = 2 \times 10^{-4}, \quad \frac{\bar{S}_{ur}}{\sqrt{\bar{u}_r^2 S^2}} = 3 \times 10^{-3}
\]

**Poloidal \( g \)-mode A1**

\[
\frac{\bar{u}_D^2}{\bar{u}_T^2} = 43, \quad \frac{\bar{S}_{ur}}{\sqrt{\bar{u}_r^2 S^2}} = 6.4 \times 10^{-6}
\]

![Toroidal Flow Stream Lines](image1)

![Radial Velocity](image2)

![Entropy](image3)

![Toroidal Flow Stream Lines](image4)

![Radial Velocity](image5)

![Entropy](image6)
Possibility of dynamos

$$H_{\text{rel}} = \frac{\langle u \cdot (\nabla \times u) \rangle}{k u^2}$$

Alecian et al. (2013) observed rapid (~10 yrs) changes in global magnetic field of Herbig star HD 190073.
Even a very small radial differential rotation can provoke baroclinic instability in stellar radiation zones.

The instability can be understood as stability loss to excitation of $r$- and $g$-modes of global oscillations.

The unstable disturbances are helical. Turbulence resulting from the instability can be prone to dynamos.