Non-Dissipative Saturation of the Magneto-Rotational Instability

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1 Introduction

2 The Reduced MHD Equations
   - The Thin Disk Approximation
   - The Reduced Equations

3 The Linear Problem
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Accretion Disks

- Turbulence needed to account for angular momentum transfer outwards and excess in infra red radiation.
- What is the source of turbulence in accretion disks?
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- What is the source of turbulence in accretion disks?
The Magnetorotational Instability (MRI)

MRI Wave Pattern

- Keplerian rotation.
- Infinite cylinder.
- Axisymmetric perturbations.

Alfvén-Coriolis waves

MRI

Alfvén waves (no rotation)

Coriolis (epicyclic) oscillations
Dissipative Saturation

- Magnetorotational Instability (MRI) [Velikhov (1959), Chandrasekhar (1960)]. Reintroduced by Balbus and Hawley (1991) as a major source of turbulence in thin astrophysical disks.

- Knobloch and Julien (2005) demonstrated saturation of the MRI far from threshold in infinite axially uniform cylindrical plasmas with rigid walls.

- Umurhan et al. (2007) employed the shearing box description in order to show that near threshold the MRI saturation level decreases with the magnetic Prandtl number.

\[ A_s \to \sqrt{P_m}, \quad P_m \to 0 : \quad \dot{L} \to 1/Re, \quad Re \to \infty \]

- We consider a new dynamical process: non-dissipative saturation in axially stratified thin disks.
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Thin Disk Geometry

- Axial stretching: $\zeta = \frac{z}{\epsilon}$, $\epsilon = \frac{H}{R}$, $\frac{\partial}{\partial z} = \frac{1}{\epsilon} \frac{\partial}{\partial \zeta}$.

- Supersonic rotation: Rotation Mach Number $= \frac{1}{\epsilon}$.

- Radial force balance: $v_\theta = r\Omega(r)$.

- Axial force balance: $\rho(r, \zeta) = \rho_0(r)e^{-\zeta^2/2H(r)^2}$.

- Free functions: $B_z(r), \rho_0(r), T(r)$.
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Thin Disk Geometry

**Important Parameters**

- Plasma beta:
  \[ \beta(r) = \beta_0 \frac{\rho_0(r)T(r)}{B_z^2(r)}. \]

- Disk semi-thickness:
  \[ H(r) = \frac{\sqrt{T(r)}}{\Omega(r)}. \]
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Axial Density Stratification

- \[ \eta = \frac{\zeta}{H(r)} \]
- \[ \rho / \rho_0 = e^{-\eta^2 / 2} \]
- \[ \rho / \rho_0 = sech^2(\eta) \]
The Reduced Thin-Disk MHD Equations

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{bmatrix} x_{ac} \\ x_{ms} \end{bmatrix} &= \begin{bmatrix} \mathcal{L}_{ac}(\eta) & 0 \\ 0 & \mathcal{L}_{ms}(\eta) \end{bmatrix} \begin{bmatrix} x_{ac} \\ x_{ms} \end{bmatrix} + \begin{bmatrix} N_{ac}(x_{ac}, x_{ms}) \\ N_{ms}(x_{ac}, x_{ms}) \end{bmatrix} \\
\end{align*}
\]

\[x_{ac} \equiv \begin{bmatrix} v_r \\ v_\theta \\ b_r \\ b_\theta \end{bmatrix}\]

Alfvén-Coriolis in plane perturbations

\[x_{ms} = \begin{bmatrix} v_z \\ \sigma \end{bmatrix}\]

Magneto-Sonic vertical perturbations
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**Alfven-Coriolis in plane perturbations**

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The Alfvén-Coriolis Spectrum

- Analytical solution for the $\rho / \rho_0 = sech^2(\eta)$ vertical profile.

- Assuming that the perturbations evolve in time as $e^{i\omega \Omega t}$:

$$
(L_{ac} + K^+)(L_{ac} + K^-)V_{\theta,r} = 0
$$

$$
L_{ac} = \frac{d}{d\zeta} [(1 - \zeta^2) \frac{d}{d\zeta}], \quad \zeta = tanh(\eta), \quad \text{Legendre operator}
$$

$$
K^\pm = \frac{\pi \beta(r)}{4} (3 + 2\omega^2 \pm \sqrt{9 + 16\omega^2})
$$

- Solutions diverge at most polynomial at $\eta \to \pm\infty$ if:

$$
K^\pm = k(k + 1), \quad k = 1, 2, \ldots \quad \text{and} \quad v_{\theta,r} = P_k[tanh(\eta)]
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The Alfvén-Coriolis Eigenfunctions

$k = 2$  $k = 3$  $k = 4$
The Alfvén-Coriolis Dispersion Relation - MRI

\[ K^\pm = k(k + 1) \]

\[ \downarrow \]

\[ (3\beta_{cr}^k - \omega^2\beta)[3\beta_{cr}^k - (3 + \omega^2)\beta] - 4\omega^2\beta^2 = 0 \]

\[ \beta_{cr}^k \equiv \frac{k(k + 1)}{3} \]

\( k \) unstable MRI modes for \( \beta(r) > \beta_{cr}^k \).
The Alfvén-Coriolis Dispersion Relation - MRI

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The Magnetosonic System

MRI Stability Bifurcation Plot

\[ \gamma = \text{Im} \omega \]

\[ \beta_{cr}^k = \frac{k(k+1)}{3} \]

Zero eigenvalue of multiplicity two at each bifurcation point
The Stable Magnetosonic Spectrum

- Analytical solution for the $\rho/\rho_0 = sech^2(\eta)$ vertical profile.

- Assuming that the perturbations evolve in time as $e^{i\omega \Omega t}$:

$$
(1 - \xi^2) \frac{d^2\sigma}{d\xi^2} + \left[ \frac{\omega^2}{1 - \xi^2} + 2 \right] \sigma = 0, \quad \xi = \tanh(\eta)
$$

Boundary conditions: $\sigma(\eta) \to 0$ at $\eta \to \pm \infty$ ($\xi \to \pm 1$)

- Useful substitution:

$$
\sigma(\xi) = \sqrt{1 - \xi^2} f(\xi)
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The Stable Magnetosonic Spectrum

- The associated Legendre equations

\[
(1 - \xi^2) \frac{d^2f}{d\xi^2} - 2\xi \frac{df}{d\xi} + \left[ 2 - \frac{\mu^2}{1 - \xi^2} \right] = 0
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- Solution

\[
\sigma(\eta) = \sqrt{1 - \xi^2} \left[ a_+ f_+(\xi) + a_- f_-(\xi) \right]
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f_{\pm} = \left[ \frac{1 - \xi}{1 + \xi} \right]^{\pm\mu/2} (\mu \pm \xi)
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\]
Solutions exist for $\omega^2 > 0$ hence the magnetosonic modes are stable and have a continuous spectrum.
Consider $\beta$ values slightly above the threshold of the first unstable mode ($k = 1$):

$$\beta = \beta_{cr}^1 + \delta$$

$\delta$ is a control parameter that is related to the growth rate as:

$$\gamma^2 = \frac{27\delta}{14}$$

Express any perturbation $f$ as:

$$f(r, \eta, t) = \phi_1(\eta)a(t)$$

For small perturbations the amplitude is:

$$a(t) = a_0 e^{\gamma(\delta)t}.$$ 

Goal of weakly nonlinear analysis: to find differential equation in time for $a(t)$. 
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The Amplitude Equation

- Transition to instability ($\delta = 0$) through double zero eigenvalue.

- Crossing the first instability threshold:
  - The single stable fixed point at the center turns into a (unstable) saddle.
  - Two extra stable fixed points emerge.
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\[ \delta < 0 \quad \quad \quad \delta > 0 \]

Duffing equation: \[ \frac{d^2a}{dt^2} = \gamma^2 a - \alpha a^3 \]
How to Calculate $\alpha$?

1. Find the new stable steady-state of the reduced MHD equations:

   - $v_r = v_z = b_\theta = 0$, \( \frac{\partial}{\partial t} (v_\theta, b_r, \sigma) = 0 \)
   - $b_r(\eta) = \sqrt{\delta} \mu_1 \phi_1(\eta) + (\sqrt{\delta})^3 \mu_3 \phi_3(\eta) + \ldots$

2. The equation for $\phi_1$ is obtained from lowest order and is:

   $$\mathcal{L}(\phi_1) = 0$$

   where $\mathcal{L}$ is the linear Alfvén-Coriolis operator.
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where \( \mathcal{L} \) is the linear Alfvén-Coriolis operator.
How to Calculate $\alpha$?

3. Solution of lowest order linear equation:

$$\phi_1(\eta) = P_0(\xi) - \xi P_1(\xi)$$

4. Next order equation:

$$\mathcal{L}(\phi_3) = \mathcal{N}(\mu_1 \phi_1)$$

5. Solvability condition for $\phi_3$:

$$\langle \phi_1 \mathcal{N}(\mu_1 \phi_1) \rangle = 0$$

Result:

$$\mu_1 = \sqrt{5/2}$$
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3. Solution of lowest order linear equation:

$$\phi_1(\eta) = P_0(\xi) - \xi P_1(\xi)$$

4. Next order equation:

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5. Solvability condition for $\phi_3$:

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Numerical Calculations

Full line: Duffing Equation

Dashed Line: Nonlinear MHD Equations
The Physical Mechanism

- The growing perturbed magnetic pressure pushes the plasma away and reduces mid-plane density.
- The Alfvén velocity increases.
- The beta value decreases below threshold.
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One conservation law for the amplitude:

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\left( \frac{da}{dt} \right)^2 + \frac{1}{2} \alpha a^4 - \gamma^2 a^2 = h
\]

\(a(t)\) is expressed in terms of the Jacobi Elliptic Functions.
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\Rightarrow \quad a(t) \text{ is bounded.}
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Sensitivity of the period of the nonlinear oscillations to Initial Conditions:

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P \to \frac{1}{2\gamma} \ln \left( \frac{32\gamma^2}{\alpha h} \right) \quad \text{as} \quad h \to 0
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Properties of the Weakly Nonlinear Solution

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Summary

- A new non-dissipative saturation mechanism of the MRI has been demonstrated.

- The MRI excites Magneto-Sonic waves that modify the plasma density.

- The MRI saturates in form of bursty oscillations that drive the system in and out of the stable regime.

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For Further Reading I

- E. Liverts and M. Mond
  *MNRAS*, 2(1):50–100,

- Y. Shtemler, M. Mond, and E. Liverts
  *MNRAS*, 2(1):50–100,

- E. Knobloch and K. Julien
  *MNRAS*, 2(1):50–100,

- O.M. Umurhan, K. Menou, and O. Regev
  *MNRAS*, 2(1):50–100,

- E. Liverts, Y. Shtemler, M. Mond, O.M. Umurhan, and D. Bisikalo
  *Phys. Rev. Lett.*, 2(1):50–100,
For Further Reading II

O.M. Umurhan and O.Regev.

*Fluid Dynamics.*

Coming soon to a library near you, 2013.