Handling Handles: Non-Planar AdS/CFT Integrability

Part 2

(Part 1 by J. Caetano)

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+ further work in progress

Workshop on Correlation Functions in Solvable Models

Nordita, Stockholm, May 2018

Non-Planar Processes: Idea

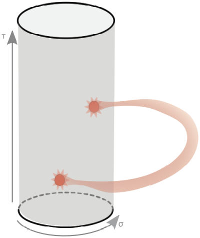
Hexagonalization: Works for planar (4,5)-point functions [Fleury '16 Komatsu] [Fleury '17 Komatsu]

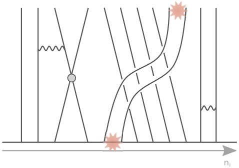
Extend to non-planar processes?

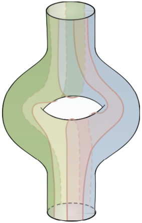
- Fix worldsheet topology
- Dissect into planar hexagons
- Glue hexagons (mirror states)

Simple Proposal:

$$\left\langle \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle^{\text{full}} = \frac{1}{N_c^{n-2}} \sum_g \frac{1}{N_c^{2g}} \sum_{\substack{\text{graphs} \\ (\text{genus } g)}} \prod_c d_c^{\ell_c} \sum_{\substack{\text{mirror} \\ \text{states}}} \mathcal{H}_1 \, \mathcal{H}_2 \, \mathcal{H}_3 \ldots \mathcal{H}_F$$







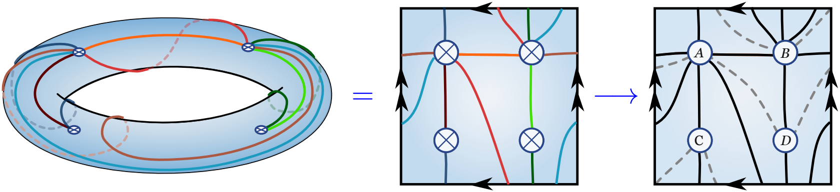
Sum over Graphs: Cutting the Torus

Sum over propagator graphs: Split into

- Sum over graphs with non-parallel edges ($\equiv$ "bridges")
- Sum over distributions of parallel propagators on bridges

Torus with four punctures: *How many hexagons/bridges?*
Euler: $F + V - E = 2 - 2g$.
Our case: $g = 1$, $V = 4$, $E = \frac{3}{2}F \quad \Rightarrow \quad F = 8$, $E = 12$.

$\rightarrow$ Construct all genus-one graphs with 4 punctures and up to 12 edges.



Propagators may populate < 12 bridges and still form a genus-one graph.
Such graphs will contain higher polygons besides hexagons.
$\rightarrow$ Subdivide into hexagons by inserting zero-length bridges (ZLBs)

Maximal Graphs

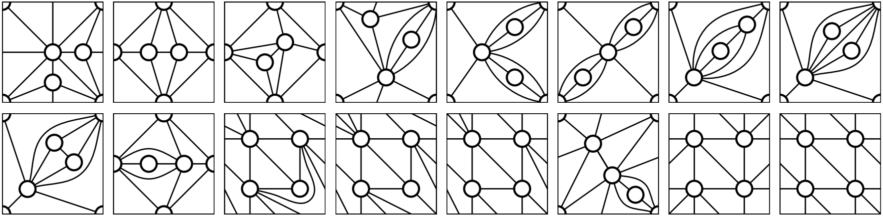
Focus on Maximal Graphs: Graphs with a maximal number of edges.

- Adding any further edge would increase the genus
- Maximal graphs ⇔ triangulations of the torus.

Construction:

- Manually: Add one operator at a time, in all possible ways.
- Computer algorithm: Start with the empty graph, add one bridge in all possible ways, iterate. → Systematic.

Complete list of maximal graphs:



Submaximal Graphs

Submaximal graphs: Graphs with a non-maximal number of edges.

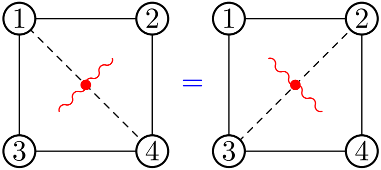
- Obtained from maximal graphs by deleting bridges.
- Number of genus-one graphs by number of bridges:

#bridges:	12	11	10	9	8	7	6	5	$\leq$4
#graphs:	7	28	117	254	323	222	79	11	0

Hexagonalization:

Submaximal graphs contain higher polygons (octagons, decagons, . . .).

- Must be subdivided into hexagons by zero-length bridges.
- Subdivision is not physical: Can pick any (flip invariance): [Fleury '16 Komatsu]

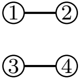


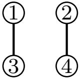
The Data: Kinematics

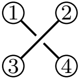
Half-BPS operators:

$$\mathcal{Q}_i^k \equiv \mathrm{Tr}\big[(\alpha_i \cdot \Phi(x_i))^k\big]\,, \quad \Phi = (\phi_1, \ldots, \phi_6)\,, \quad \alpha_i^2 = 0\,.$$

For equal weights (k,k,k,k): Expand in X, Y, Z:

$$X \equiv \frac{\alpha_1 \cdot \alpha_2\, \alpha_3 \cdot \alpha_4}{x_{12}^2 x_{34}^2} =$$


$$,\quad Y \equiv$$


$$,\quad Z \equiv$$


$$.$$

Focus on $Z = 0$ (polarizations): [Arutyunov Sokatchev '03] [Arutyunov,Penati '03 Santambrogio,Sokatchev]

$$G_k \equiv \langle \mathcal{Q}_1^k \mathcal{Q}_2^k \mathcal{Q}_3^k \mathcal{Q}_4^k \rangle^{\text{loops}} = R \sum_{m=0}^{k-2} \mathcal{F}_{k,m}\, X^m Y^{k-2-m}$$

Supersymmetry factor: $R = z\bar{z}X^2 - (z+\bar{z})XY + Y^2$

Main data: Coefficients $\mathcal{F}_{k,m} = \mathcal{F}_{k,m}(g; z, \bar{z})$

Cross ratios: $z\bar{z} = s = \dfrac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad (1-z)(1-\bar{z}) = t = \dfrac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}\,.$

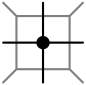
The Data: Quantum Coefficients

Data Functions: Correlator coefficients:

$$\mathcal{F}_{k,m} = \sum_{\ell=1}^{\infty} g^{2\ell} \mathcal{F}^{(\ell)}_{k,m}(z,\bar{z})\,, \quad \text{'t Hooft coupling: } g^2 = \frac{g^2_{\mathrm{YM}} N_{\mathrm{c}}}{16\pi^2}\,.$$

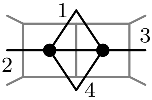
One and two loops: Two ingredients: Box integrals

$$F^{(1)}(z,\bar{z}) = \frac{x^2_{13}x^2_{24}}{\pi^2} \int \frac{\mathrm{d}^4 x_5}{x^2_{15}x^2_{25}x^2_{35}x^2_{45}} =$$



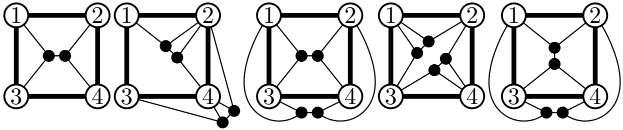
,

$$\frac{F^{(2)}(z,\bar{z})}{x^2_{14}} = \frac{x^2_{13}x^2_{24}}{(\pi^2)^2} \int \frac{\mathrm{d}^4 x_5\, \mathrm{d}^4 x_6}{x^2_{15}x^2_{25}x^2_{45}x^2_{56}x^2_{16}x^2_{36}x^2_{46}} =$$



,

& Color factors: $C^{i}_{k,m} \in \{$



$\}$

① $= \mathrm{Tr}(T^{(a_1} \ldots T^{a_k)})$,

$= f_{ab}{}^{c}$

The Data: Color Factors

To obtain non-planar corrections: Need to expand color factors.

$$C^i_{k,m} = N_c^{2k} k^4 \left({}^{\bullet}C^i_{k,m} + {}^{\circ}C^i_{k,m} N_c^{-2} + \mathcal{O}(N_c^{-4}) \right) \,, \quad i \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\} \,,$$

Compute by brute force:

k	m	$\frac{1}{2}{}^{\circ}C^{1,\mathrm{U}}_{k,m}$	$\frac{1}{2}{}^{\circ}C^{1,\mathrm{SU}}_{k,m}$	${}^{\circ}C^{\mathrm{a,U}}_{k,m}$	$2{}^{\circ}C^{\mathrm{b,U}}_{k,m}$	$\frac{1}{2}{}^{\circ}C^{\mathrm{c,U}}_{k,m}$	${}^{\circ}C^{\mathrm{d,U}}_{k,m}$	${}^{\circ}C^{\mathrm{a,SU}}_{k,m}$	$2{}^{\circ}C^{\mathrm{b,SU}}_{k,m}$	$\frac{1}{2}{}^{\circ}C^{\mathrm{c,SU}}_{k,m}$	${}^{\circ}C^{\mathrm{d,SU}}_{k,m}$
2	0	1	1	0	−2	−1	−1	0	−2	−1	−1
3	0	1	9	−5	−2	−1	−1	−9	−18	−9	−9
3	1	1	9	0	3	−1	−1	0	−5	−9	−9
4	0	−5	13	−7	10	5	5	−25	−26	−13	−13
4	1	−12	24	4	15	13	14	−23	−21	−23	−22
4	2	−5	13	0	21	5	5	0	3	−13	−13
5	0	−23	9	−1	46	23	23	−33	−18	−9	−9
5	1	−51	13	31	47	55	59	−33	−17	−9	−5
5	2	−51	13	39	76	55	59	−9	12	−9	−5
5	3	−23	9	0	63	23	23	0	31	−9	−9
6	0	−61	−11	20	122	61	61	−30	22	11	11
6	1	−126	−26	92	107	135	144	−8	7	35	44
6	2	−159	−59	139	187	175	191	39	87	75	91
6	3	−126	−26	110	201	135	144	35	101	35	44
6	4	−61	−11	0	139	61	61	0	89	11	11

also: $k = 7, 8, 9$. All color factors are quartic polynomials in m and k.

The Data: Result

$$\mathcal{F}^{(1),\mathrm{U}}_{k,m}(z,\bar z) =$$

$$-\frac{2k^2}{N_c^2}\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\left[\tfrac{9}{2}r^2-\tfrac{13}{8}\right]k^3+\left[\tfrac{1}{6}r^2+\tfrac{15}{8}\right]k^2-\tfrac{1}{2}k\right]\right\}F^{(1)},$$

$$\mathcal{F}^{(2),\mathrm{U}}_{k,m}(z,\bar z) =$$

$$\frac{4k^2}{N_c^2}\Bigg[\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\left[\tfrac{9}{2}r^2-\tfrac{13}{8}\right]k^3+\left[\tfrac{1}{6}r^2+\tfrac{15}{8}\right]k^2-\tfrac{1}{2}k\right]\right\}F^{(2)}$$

$$+\left\{\frac{t}{4}+\frac{1}{N_c^2}\left[\left(\left[\tfrac{7}{2}r^2-\tfrac{1}{8}\right]k^2+\tfrac{5}{8}k-\tfrac{1}{4}\right)s_+-r\left(\left[\tfrac{17}{6}r^2-\tfrac{7}{8}\right]k^3+3k^2-\tfrac{13}{12}k\right)s_-\right.\right.$$

$$\left.\left.+\left(\left[\tfrac{29}{24}r^4-\tfrac{11}{16}r^2+\tfrac{15}{128}\right]k^4+\left[\tfrac{17}{8}r^2-\tfrac{21}{32}\right]k^3-\left[\tfrac{23}{24}r^2-\tfrac{39}{32}\right]k^2-\tfrac{9}{8}k+\tfrac{1}{2}\right)t\right]\right\}\left(F^{(1)}\right)^2$$

$$-\frac{1}{N_c^2}\left[r\left\{\left[\tfrac{7}{6}r^2-\tfrac{1}{8}\right]k^3+\tfrac{3}{2}k^2+\tfrac{10}{3}k\right\}F^{(2)}_{\mathrm{C},-}\right.$$

$$\left.+\left\{\left[\tfrac{5}{4}r^2-\tfrac{19}{48}\right]k^3+\left[\tfrac{3}{2}r^2+\tfrac{7}{8}\right]k^2+\tfrac{1}{3}k\right\}F^{(2)}_{\mathrm{C},+}\right]$$

$$+\frac{1}{4}\left\{1+\frac{(k-1)(k^3+3k^2-46k+36)}{12N_c^2}\right\}\left(s\delta_{m,0}+\delta_{m,k-2}\right)\left(F^{(1)}\right)^2$$

$$+\left\{1+\frac{(k-2)_4}{12N_c^2}\right\}\left(\delta_{m,0}F^{(2)}_{z-1}+\delta_{m,k-2}F^{(2)}_{1-z}\right)\Bigg],$$

where $r=(m+1)/k-1/2$. $\mathcal{F}_{k,m}$: Coefficient of X^mY^{k-2-m}.

First Test: Large k: Data and Graphs

Focus on leading order in large k → several simplifications:

Data:

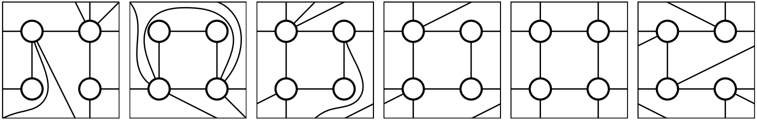
$$\mathcal{F}^{(1),\mathrm{U}}_{k,m}(z,\bar{z}) = -\frac{2k^2}{N_c^2}\Big\{1+\frac{1}{N_c^2}\Big[\big[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\big]k^4+\mathcal{O}(k^3)\Big]\Big\}F^{(1)}\,,$$

$$\mathcal{F}^{(2),\mathrm{U}}_{k,m}(z,\bar{z}) = \frac{4k^2}{N_c^2}\Bigg[\Big\{1+\frac{1}{N_c^2}\Big[\big[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\big]k^4+\mathcal{O}(k^3)\Big]\Big\}F^{(2)} + \Big\{1+\frac{1}{N_c^2}\Big[\big[\tfrac{29}{6}r^4-\tfrac{11}{4}r^2+\tfrac{15}{32}\big]k^4+\mathcal{O}(k^3)\Big]\Big\}\frac{t}{4}\big(F^{(1)}\big)^2\Bigg]\,.$$

Combinatorics of distributing propagators on bridges:
Sum over distributions of m propagators on $j+1$ bridges → $m^j/j!$

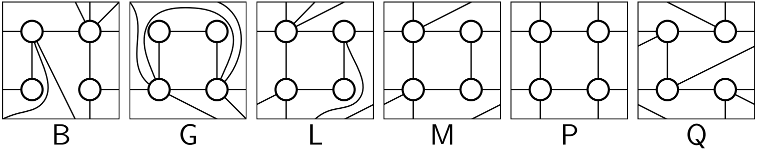
- ⇒ Only graphs with maximum bridge number contribute.
- ⇒ All bridges carry a large number of propagators.

Graphs: $(Z=0)$



First Test: Large k: Graphs and Labelings

Graphs:

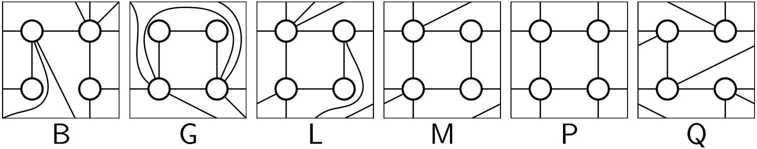


Sum over labelings:

Case	Inequivalent Labelings (clockwise)	Combinatorial Factor
B	$(1,2,4,3),(2,1,3,4),(3,4,2,1),(4,3,1,2)$	$m^3(k-m)/6$
B	$(1,3,4,2),(3,1,2,4),(2,4,3,1),(4,2,1,3)$	$m(k-m)^3/6$
G	$(1,2,4,3),(3,4,2,1)$	$m^4/24$
G	$(1,3,4,2),(2,4,3,1)$	$(k-m)^4/24$
L	$(1,2,4,3),(3,4,2,1),(2,1,3,4),(4,3,1,2)$	$m^2/2\cdot(k-m)^2/2$
M	$(1,2,4,3),(2,1,3,4),(1,3,4,2),(3,1,2,4)$	$m^2(k-m)^2/2$
P	$(1,2,4,3)$	$m^2(k-m)^2/2$
Q	$(1,2,4,3)$	$m^2(k-m)^2$

First Test: Large k: Octagons

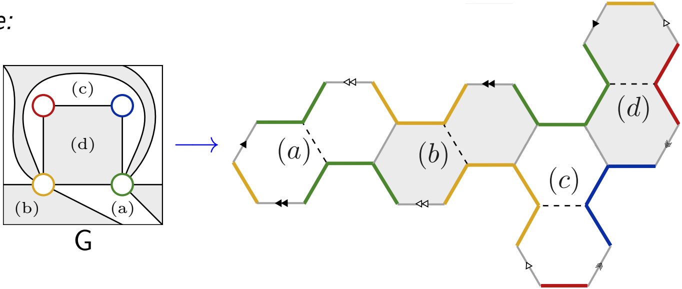
Graphs:



All graphs consist of only octagons!

Split each octagon into two hexagons with a zero-length bridge.

Example:



First Test: Large k: Mirror Particles

Loop Counting:

Expand mirror measure $\mu(u) \sim e^{-\ell \tilde{E}(u)}$ and hexagons $\mathcal{H}$ in coupling g
$\rightarrow$ n particles on bridge of size ℓ: $\mathcal{O}(g^{2(n\ell+n^2)})$

All graphs consist of octagons framed by parametrically large bridges
$\rightarrow$ Only excitations on zero-length bridges inside octagons survive

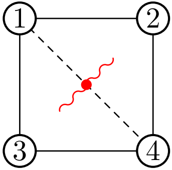
Excited Octagons:

n particles on a zero-length bridge $\rightarrow$ $\mathcal{O}(g^{2n^2})$
$\rightarrow$ Octagons with $1/2/3/4$ particles start at $1/4/9/16$ loops

Octagon 1–2–4–3 with 1 particle:

[Fleury '16 Komatsu] [TB,Caetano,Fleury Komatsu,Vieira '18]

$$\mathcal{M}(z,\alpha) = \left[z + \bar{z} - (\alpha + \bar{\alpha})\frac{\alpha\bar{\alpha} + z\bar{z}}{2\alpha\bar{\alpha}}\right] \cdot \left(g^2 F^{(1)}(z) - 2g^4 F^{(2)}(z) + 3g^6 F^{(3)}(z) + \ldots\right)$$



For $Z = 0$: R-charge cross ratios
$\alpha = z\bar{z}\, X/Y$ and $\bar{\alpha} = 1$.

First Test: Large k: Match and Prediction

We are Done:

Sum over graph topologies and labelings (with bridge sum factors),
Sum over one-particle excitations of all octagons.
$\Rightarrow$ Result matches data and produces prediction for higher loops!

Summing all octagons gives:

$$\mathcal{F}^{\mathrm{U}}_{k,m}(z,\bar{z})\big|_{\text{torus}} = -\frac{2k^6}{N_c^4}\bigg\{$$

$$g^2\left[\tfrac{17}{6}r^4 - \tfrac{7}{4}r^2 + \tfrac{11}{32}\right]F^{(1)} \quad \checkmark \text{ match}$$

$$-\,2g^4\left[\left[\tfrac{17}{6}r^4 - \tfrac{7}{4}r^2 + \tfrac{11}{32}\right]F^{(2)} + \left[\tfrac{29}{6}r^4 - \tfrac{11}{4}r^2 + \tfrac{15}{32}\right]\frac{t}{4}\left(F^{(1)}\right)^2\right] \quad \checkmark \text{ match}$$

$$+\,g^6\left[\left[\ldots\right]F^{(3)} + \left[\ldots\right]\left(F^{(2)}\right)\left(F^{(1)}\right) + \left[\ldots\right]\left(F^{(1)}\right)^3\right] \quad \text{prediction!}$$

$$+\,\mathcal{O}(g^8) + \mathcal{O}(1/k)\bigg\}\,.$$

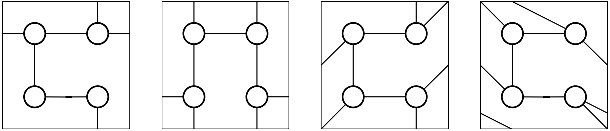
More Tests: $k = 2, 3, 4, 5, \ldots$

Small and finite k:

Few propagators → Fewer bridges → Graphs with fewer edges

⇒ Graphs composed of not only octagons, but bigger polygons

Example: Graphs for $k = 3$:



Hexagonalization:

Each $2n$-gon: Split into $n - 2$ hexagons by $n - 3$ zero-length briges.

Loop Expansion: Much more complicated!

All kinds of excitation patterns already at low loop orders

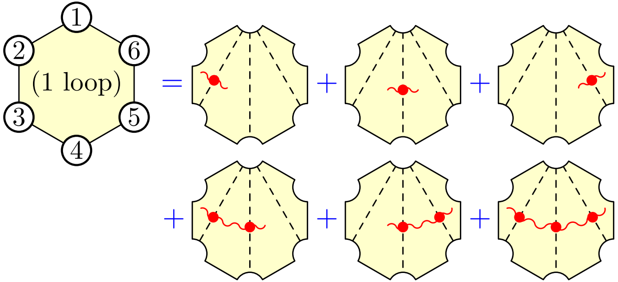
- Single particles on several adjacent zero-length (or $\ell = 1$) bridges
- Strings of excitations wrapping around operators

Finite k: One Loop: Sum over ZLB-Strings

Restrict to one loop: Only single particles on one or more adjacent zero-length bridges contribute.

$\Rightarrow$ Excitations confined to single polygons bounded by propagators.

For each polygon: Sum over all possible one-loop strings:

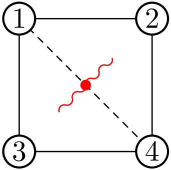


One-strings: understood ✓

Longer strings: need to compute!

Two-String: Result

One-String: Can be written as

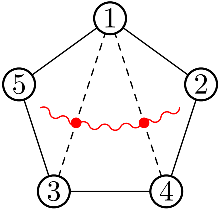


$$\mathcal{M}^{(1)}(z,\alpha) = m(z) + m(z^{-1})\,,$$

with building block

$$m(z) = m(z,\alpha) = g^2\,\frac{(z+\bar{z})-(\alpha+\bar{\alpha})}{2}\,F^{(1)}(z,\bar{z})$$

Two-string: Despite complicated computation, simplifies to [Fleury '17 Komatsu]



$$\begin{aligned}
&\mathcal{M}^{(2)}(z_1,z_2,\alpha_1,\alpha_2) \\
&= m\left(\frac{z_1-1}{z_1 z_2}\right) + m\left(\frac{1-z_1+z_1 z_2}{z_2}\right) \\
&\quad + m(z_1(1-z_2)) - m(z_1) - m(z_2^{-1})\,,
\end{aligned}$$

with the same building block $m(z)$!

Finite k: Results

Done! Sum over all graphs, expand all polygons to their one-loop values.

Numbers of labeled graphs with assigned bridge sizes:

k:	2	3	4	5
$g=0$:	3	8	15	24
$g=1$:	0	32	441	2760

Result: For $k = 2, 3, 4, 5, \ldots$:
Matches the $\mathrm{U}(N_\mathrm{c})$ data $\mathcal{F}_{k,m}$, up to a copy of the planar term!

$$\mathcal{F}_{k,m}: \quad \texttt{Result} = (\texttt{torus data}) + \frac{1}{N_\mathrm{c}^2}(\texttt{planar data})$$

✓✓✓ (torus data) ??? (planar data)

What does this mean?? $\Rightarrow$ Puzzle.

Difference between $\mathrm{U}(N_\mathrm{c})$ and $\mathrm{SU}(N_\mathrm{c})$? $\rightarrow$ No
Operator normalizations? $\rightarrow$ No
Need to include planar graphs on the torus? If yes, how?

Finite k: Stratification

We are computing a worldsheet process.
The string amplitude involves integration over moduli space $\mathcal{M}_{g,n}$.

Sum over graphs: Reminiscent of moduli space integration.
This can be made more precise:
Moduli space $\Leftrightarrow$ space of *metric ribbon graphs* $\mathrm{RGB}^{\mathrm{met}}_{g,n}$.

Metric Ribbon Graphs with labeled Boundary:
Regular graphs, but edges at each vertex have definite ordering.
Double-line notation defines n oriented boundary components (faces).
Faces define compact oriented surface of definite genus g.
Assign length $\ell_j \in \mathbb{R}_+$ to each edge.

Bijection: Via Strebel theory:

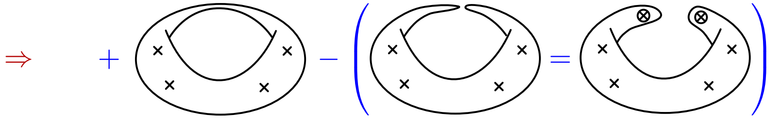
$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \quad \longleftrightarrow \quad \mathrm{RGB}^{\mathrm{met}}_{g,n} = \coprod_{\Gamma \in \mathrm{RG}_{g,n}} \frac{\mathbb{R}_+^{e(\Gamma)}}{\mathrm{Aut}_\partial(\Gamma)}$$

Finite k: Stratification

Discretization: Need to be careful at the boundaries of the space. Do not overcount/undercount. Boundary of torus moduli space: All bridges traversing a handle reduce to zero size $\longrightarrow$ handle gets pinched.

This problem has been considered before in the context of matrix models. [Deligne Mumford '69] [Chekhov 1995]

Resolution: In the sum over graphs, include planar graphs drawn on the torus. This leads to some overcounting. Compensate by subtracting planar graphs with two extra fictitious zero-size operators. *Stratification*.



Including these contributions indeed accounts for the $(\text{planar})/N_c^2$ term!

$\Rightarrow$ Now have a complete match for $k = 2, 3, 4, 5$.

Summary & Outlook

Summary: Method to compute higher-genus terms in $1/N_{\text{c}}$ expansion.

- Sum over free graphs, decompose into planar hexagons, integrate over mirror states.
- Large k: Only octagons, match at two loops, three-loop prediction
- Match for various finite $k \to$ stratification

Outlook: There are many things to do that we currently explore:

- Study more examples: Higher loops / genus, more general operators
- Understand details/implications of stratification beyond one loop
- Connect to recent supergravity loop computations at strong coupling? [Aharony, Alday, Bissi, Perlmutter '16] [Alday, Bissi, Perlmutter '17] [Alday, Bissi '17] [Aprile, Drummond, Heslop, Paul '17, '17, '17, '18]
- Promising: Large k at higher genus: Only octagons. Speculate: Resum $1/N_{\text{c}}$?