“Explosive Percolation”

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R. A. da Costa, SND, A. V. Goltsev, and J. F. F. Mendes,
“Explosive percolation” transition is actually continuous,
arXiv:1009.2534

simulation: 512,000 nodes

ordinary percolation: “explosive percolation”:

\[ \min(s_1 s_2, s_3 s_4) \]
How can it be?: discontinuity coexisting with critical power-law distributions and scaling above and below this transition - ???
Representative model of “explosive percolation”

ordinary percolation:

“explosive percolation” \((m = 2)\):

A

B

(arXiv:1009.2534)
Relation between models
Simulations $2 \times 10^9$ nodes:

$$S \propto \delta^\beta$$

$$\beta \approx 0.0476 \approx 1/18, \ \delta = t - t_c$$
Estimate

Suppose $\beta = 1/18$ and $N = 10^{18}$. The smallest time interval is $1/N$. Then a single step from the percolation threshold gives

$$S \sim (10^{-18})^{1/18} \sim 0.1$$

So simulations are virtually useless. We must study the infinite system.
Distributions

\( n(s) \) and \( P(s) \) are for clusters

\[
P(s) = sn(s)/\langle s \rangle,
\]

\[
\sum_s P(s) = 1 - S
\]

\( Q(s) \) is for merging clusters,

\[
\sum_s Q(s) = 1 - S^2
\]

\[
Q_{\text{cum}}(s) + S^2 = [P_{\text{cum}}(s) + S]^2
\]

\[
Q(s) = [P_{\text{cum}}(s) + P_{\text{cum}}(s + 1) + 2S]P(s)
\]

\[
= [2 - 2P(1) - 2P(2) - \ldots - 2P(s-1) - P(s)]P(s)
\]
Equations

\[
\frac{\partial P(s, t)}{\partial t} = s \sum_{u+v=s} Q(u, t)Q(v, t) - 2sQ(s, t)
\]

exactly describe the evolution of the distributions in the full range of \( t \) for the infinite system.

We solved numerically \( 10^6 \) equations, which gives precise description of the distributions for \( s \leq 10^6 \).
Fitting by the law $S_0 + C \delta^\beta$ gives $S_0 < 0.005$. 

$$s \leq 10^6$$
\[ s \leq 10^6 \]
Scaling functions

\[ P(s, t) = s^{1-\tau} f(s^{\delta^{1/\sigma}}) \]
\[ Q(s, t) = s^{3-2\tau} g(s^{\delta^{1/\sigma}}) \]
Ordinary percolation: scaling functions

\[ P(s) = s^{1-\tau} f(s^{\delta^{1/\sigma}}) \]
Table: Percolation thresholds, critical exponents, and fractal and upper critical dimensions ($m = 2$)

<table>
<thead>
<tr>
<th></th>
<th>$t_c$</th>
<th>$\beta$</th>
<th>$\tau$</th>
<th>$\sigma$</th>
<th>$\gamma_P$</th>
<th>$\gamma_Q$</th>
<th>$d_f$</th>
<th>$d_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordinary</td>
<td>$1/2$</td>
<td>$1$</td>
<td>$5/2$</td>
<td>$1/2$</td>
<td>$1$</td>
<td>$-$</td>
<td>$4$</td>
<td>$6$</td>
</tr>
<tr>
<td>Explosive</td>
<td>$0.923207508(2)$</td>
<td>$0.0555(1)$</td>
<td>$2.04762(2)$</td>
<td>$0.857(3)$</td>
<td>$1.111(1)$</td>
<td>$1.0556(5)$</td>
<td>$2.333(1)$</td>
<td>$2.445(1)$</td>
</tr>
</tbody>
</table>

$$
\tau = 1 + \beta / (1 + 3 \beta), \quad \sigma = 1 / (1 + 3 \beta), \quad \gamma_P = 1 + 2 \beta, \\
\gamma_Q = 1 + \beta, \quad d_f = 2(1 + 3 \beta), \quad d_u = 2(1 + 4 \beta) \\
t_c(\infty) - t_c(N) \sim N^{-2/d_u}
$$

(arXiv:1009.2534)
If the distributions are power-law at the critical point, then $S \sim \delta^\beta$.
Indeed, above $t_c$, we have

$$Q(s) \approx 2SP(s)$$

at large $s$.
So the equation for the asymptotics contains only $P(s)$ and $S(t)$. This equation is very similar to that for ordinary percolation.
Using $P(s, t_c) = f(0)s^{1-\tau}$ as an initial condition, we solve this equation and find critical exponents and scaling functions above $t_c$. 

(arXiv:1009.2534)
Relation between $\tau$ and $t_c$

\[
P(s = 1, t_c) \sum_s s^{1-\tau} \approx 1
\]

\[
P(s = 1, t) = \frac{2}{1 + e^{4t}}
\]

\[
\frac{2}{1 + e^{4t_c}} \zeta(\tau - 1) \approx 1
\]

Substituting $t_c$ gives $\tau - 2 \approx 0.05$. 
With increasing $m$, $t_c$ approaches 1 and $\beta$ rapidly decreases with $m$, but the transition remains continuous.
Conclusion

There is no explosion in “explosive percolation”.

Lectures on Complex Networks