Scattering on the Coulomb branch of $\mathcal{N} = 4$ super Yang-Mills

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based on

F. Alday, J. H., J. Plefka and T. Schuster, arXiv:0908.0684 [hep-th]
J. H., S. Naculich, H. Schnitzer and M. Spradlin, arXiv:1001.1358 [hep-th]
J. H., S. Naculich, H. Schnitzer and M. Spradlin, arXiv:1004.5381 [hep-th]
+ work in progress

IGST 2010, Stockholm, July 1

Outline

Scattering on the Coulomb branch of $\mathcal{N} = 4$ SYM

- Motivation and Introduction
- Setup and one-loop example
- Extended dual conformal symmetry
- Integral basis at higher loops
- Regge limit

Scattering amplitudes in $\mathcal{N} = 4$ SYM - motivation

supersymmetric YM as a tool for QCD

1. perturbatively, the theories are very similar
   $\Rightarrow$ certain tree-level amplitudes identical in both theories
   $\Rightarrow$ at one loop, susy decomposition:
   $$A_g = (\underbrace{A_g + 4A_f + 3A_s}_{\mathcal{N}=4}) - 4(\underbrace{A_f + A_s}_{\mathcal{N}=1}) + A_s$$
2. develop and test new methods in $\mathcal{N} = 4$ SYM
   $\Rightarrow$ e.g. recursion relations for tree amplitudes, (generalized) unitarity
   $\Rightarrow$ application to QCD e.g. Blackhat, ...

Ambitious goals and prospects in $\mathcal{N} = 4$ SYM

1. Discover and understand new hidden symmetries
   (e.g. dual conformal symmetry)
2. Compute amplitudes for arbitrary number of legs and/or loops?!
3. Test AdS/CFT

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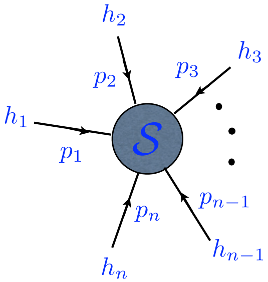
Scattering amplitudes in $\mathcal{N} = 4$ SYM

- *n*-particle scattering amplitude



  helicity: $h_i = 0$ scalar, $h_i = \pm 1$ gluon, $h_i = \pm\frac{1}{2}$ gluino

- color structure

$$A_n(\{p_i, h_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} \mathrm{tr}[t^{a_1} \ldots t^{a_n}] \times \mathcal{A}_n(\{p_{\sigma_1}, h_{\sigma_1}\}, \ldots, \{p_{\sigma_1}, h_{\sigma_1}\})$$

  $\mathcal{A}_n$: Color ordered amplitude

- IR divergences (well-understood) due to massless particles
  use e.g. dimensional regularization;
  this talk: use Higgs masses as a regulator

Reminder: Dual conformal symmetry

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu$$

- Planar MHV amplitudes have $SO(4,2)$ symmetry in the dual x_i space
  [Drummond, J.H., Smirnov, Sokatchev, 2006; Drummond, J.H., Korchemsky, Sokatchev, 2007]
- Can be extended to dual superconformal symmetry applicable to MHV and non-MHV amplitudes [Drummond, J.H., Korchemsky, Sokatchev, 2008]
- Conventional + dual superconformal $\rightarrow$ Yangian symmetry [Drummond, J.H., Plefka, 2009]

$$J^{(0)A}{}_B = \sum_i Z_i^A \frac{\partial}{\partial Z_i^B}\,, \qquad J^{(1)A}{}_B = -\sum_{i<j} Z_i^A Z_j^C \frac{\partial}{\partial Z_i^C}\frac{\partial}{\partial Z_j^B} - (i \leftrightarrow j)$$

related references: [Beisert et al, 2009+2010; Korchemsky, Sokatchev, 2009+2010, Drummond, Ferro, 2010]

$\longrightarrow$ talks by N. Beisert, L. Ferro and E. Sokatchev at this conference

Planar amplitudes on the Coulomb branch of $\mathcal{N} = 4$ Super Yang-Mills

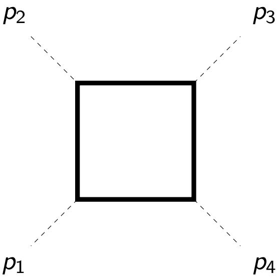
- $U(N+M) \to U(N) \times U(M)$

[Alday, Maldacena, 2007; Kawai, Suyama, 2007; Schabinger, 2008; Sever, McGreevy, 2008]

[Alday, J.H., Plefka, Schuster, 2009]

$\to$ leads to massive particles

- scatter massless $U(M)$ particles
- $N \gg M$: only allow loops in N-part of $U(N+M)$
  $\to$ renders amplitudes IR finite
- e.g. colour-ordered four-point one-loop amplitude



$$M^{(1)}(m^2/s, m^2/t)\,, \qquad s = (p_1+p_2)^2\,, \quad t = (p_2+p_3)^2$$

Various interesting limits

- Regge limit $s \gg t, m^2$

$$M^{(1)} = \log(s/m^2)\alpha(t/m^2) + O(s^0)\,, \qquad \alpha \text{ is Regge trajectory}$$

- large mass limit $m^2 \gg s, t$ [cf. Gorsky and Zhiboedov 2009]
- small mass limit $m^2 \ll s, t$ ("mass regulator")

$$M^{(1)} = -\left[\frac{1}{2}\log^2\frac{s}{m^2} + \frac{1}{2}\log^2\frac{t}{m^2}\right] + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \frac{1}{2}\pi^2 + \mathcal{O}(m^2)$$

reminder: in dimensional regularization

$$M^{(1)} = -\left[\frac{1}{\epsilon^2}\left(\frac{\mu^2}{s}\right)^{\epsilon} + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{t}\right)^{\epsilon}\right] + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \frac{2}{3}\pi^2 + \mathcal{O}(\epsilon)$$

Comments:

- geometrical interpretation as volume of tetrahedron in AdS_5

[Mason, Skinner 2010; see also Davydychev and Delbourgo 1998]

- useful in connection with momentum twistor space [Hodges 2010]

One-loop example

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Extended dual conformal symmetry

[Alday, J.H., Plefka, Schuster, 2009]

refinement: $U(N+M) \to U(N) \times U(1)^M$

- on-shell conditions now read $p_i^2 = -(m_i - m_{+1})^2$
- particles in loop have mass m_i

"extended" dual conformal symmetry

- in addition to the dual coordinates x_i^μ, we can vary the masses m_i

$$\hat{K}^\mu = K^\mu + \sum_i \left[2x_i^\mu \, m_i \frac{\partial}{\partial m_i} - m_i^2 \frac{\partial}{\partial x_{i\,\mu}} \right]$$

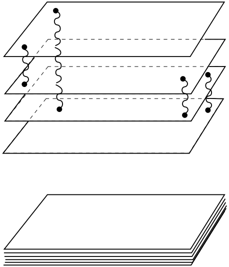
- very natural from string theory: m corresponds to radial coordinate of AdS_5
- conjecture: loop integrals have exact dual conformal symmetry

$$\hat{K}^\mu I = 0$$

String theory motivation

- string picture:

  Alday, Maldacena



(a)

- bosonic + fermionic T-duality is relevant [Alday, Maldacena, 2007; Berkovits, Maldacena, 2008]
- isometries of AdS_5 in T-dual theory

$$J_{-1,4} = m\partial_m + x^\mu \partial_\mu = \hat{D}$$
$$J_{4,\mu} - J_{-1,\mu} = \partial_\mu = \hat{P}_\mu$$
$$J_{4,\mu} + J_{-1,\mu} = 2y_\mu(x_\nu \partial^\nu + m\partial_m) - (x^2 + m^2)\partial_\mu = \hat{K}_\mu$$

- Expectation: Amplitudes regulated by Higgs masses should be invariant exactly under extended dual conformal symmetry $\hat{K}_\mu$ and $\hat{D}$! [Alday, J.H., Plefka, Schuster, 2009]

[similar ideas used in Jevicki, Kazama, Yoneda, 1998]

Properties of the Higgs regulator

Conceptual advantages

- natural from AdS/CFT viewpoint
- makes dual conformal symmetry exact
- restricts integral basis
- masses have physical interpretation

Practical advantages

- higher loop orders of amplitudes easy to compute
  e.g. $\mathcal{O}(\epsilon) \times 1/\epsilon = \mathcal{O}(1)$ problems as in dimensional regularization
- Regge limit can be computed systematically
  e.g. LL and NLL computed to all orders [J.H., Naculich, Schnitzer, Spradlin, 2010]

Implications for higher loop integral basis

- basis of loop integrals in $\mathcal{N}=4$ SYM constrained by dual conformal symmetry?

  [Drummond, J.H., Smirnov, Sokatchev, 2006; Bern, Czakon, Dixon, Kosower, Smirnov, 2006; Bern, Carrasco, Johansson, Kosower, 2007]

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  [Spradlin, Volovich, Wen, 2008]

- it seems reasonable to speculate that [J.H., Naculich, Schnitzer, Spradlin, 2010]

$$M_n = 1 + \sum_{\mathcal{I}} a^{L(\mathcal{I})}\, c(\mathcal{I})\, \mathcal{I}\,,$$

  where: coupling a, loop order $L(\mathcal{I})$
  coefficients $c(\mathcal{I}) \Rightarrow$ compute by (generalized) unitarity
  integrals $\mathcal{I} \Rightarrow$ restricted set of extended dual conformal integrals

- absence of triangles at one loop [Boels; also: Schabinger]
- additional constraints from expected IR structure [Korchemsky, Sterman,...]

$$M_n = \exp\left[-\frac{1}{8}\Gamma_{\rm cusp}(a)\sum_i \log^2\frac{s_i}{m^2} - \frac{1}{2}\tilde{G}_0(a)\sum_i \log\frac{s_i}{m^2} + \mathcal{O}(\log^0 m^2)\right]$$

- insights from analytic structure for generic m^2, and Regge limit(s)?
- further constraints from the (broken) conventional conformal symmetry?

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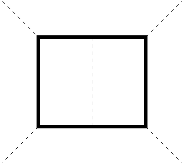
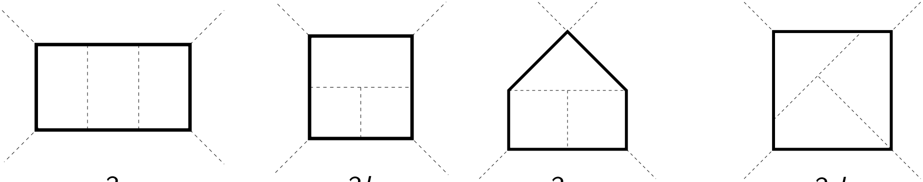
Extended dual conformal invariance at higher loops

- At 2 loops: Only one integral is allowed by extended dual conformal symmetry:



- At 3 loops: four integrals allowed:



$3a$ $3b$ $3c$ $3d$

$$M_3 = 1/(-8)\left[c_{3a}I_{3a} + c_{3b}I_{3b} + c_{3c}I_{3c} + c_{3d}I_{3d} + \{s \leftrightarrow t\}\right]$$

- Similarly restricts integral basis for more loops and legs.

Higher-loop exponentiation

- analog of Bern-Dixon-Smirnov formula [Anastasiou, Bern, Dixon, Kosower, 2002; BDS, 2003]
  in Higgs regularization: [Alday, J. H., Plefka, Schuster, 2009; J. H., Naculich, Schnitzer, Spradlin, 2010]

$$\begin{aligned} \log M_4 \;\; = \;\; & -\frac{1}{4}\Gamma_{\rm cusp}(a)\left[\log^2\frac{s}{m^2}+\log^2\frac{t}{m^2}\right]-\tilde{G}_0(a)\left[\log\frac{s}{m^2}+\log\frac{t}{m^2}\right] \\ & +\frac{1}{4}\Gamma_{\rm cusp}(a)\left[\log^2\frac{s}{t}+\pi^2\right]+\tilde{c}(a)+\mathcal{O}(m^2) \end{aligned}$$

- verified by computing dual conformal integrals up to $\mathcal{O}(m^2)$
  - at two loops [Alday, J. H., Plefka, Schuster, 2009]
  - at three loops [J. H., Naculich, Schnitzer, Spradlin, 2010]
  - at four loops (IR divergent terms and Regge limit) [J. H., Naculich, Schnitzer, Spradlin, 2010]

- method used: MB representation of all integrals, asymptotic expansion, numerical integration (Mathematica packages *MBasymptotics, MB*; *CUBA*)

- five-loop computation of $\Gamma_{\rm cusp}$ to test Beisert-Eden-Staudacher prediction

[Bourjaily, J. H., Spradlin, work in progess]

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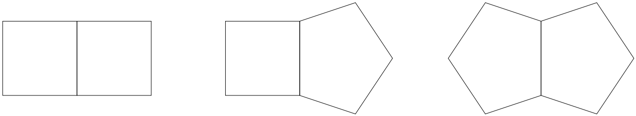
Application to two-loop integrals/amplitudes

- expected dual conformal integrals:



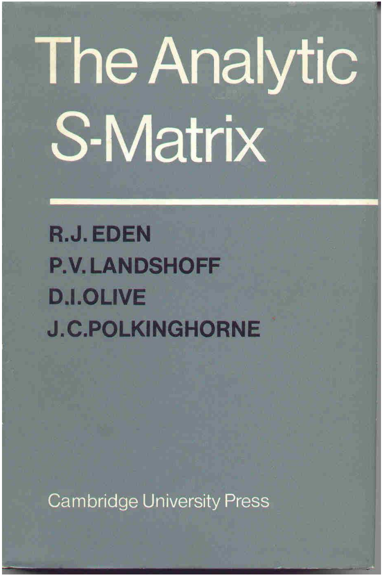
  see e.g. six-point two-loop MHV case (in dimensional regularization)

  [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich, 2008]

  and for higher-point amplitudes [Vergu, 2009]

- integrals can be evaluated straightforwardly (numerically) in mass regularization

The analytic S-matrix



Regge limits for amplitudes on the Coulomb branch

- take Regge limit $s = (p_1 + p_2)^2 \to \infty$ [J. H., Naculich, Schnitzer, Spradlin, 2010]
  expect
  $$\beta(t/m^2) \left(\frac{s}{m^2}\right)^{\alpha(t/m^2)} + \mathcal{O}(m^2)$$
  trajectory $\alpha(t/m^2) = -\frac{1}{2}\Gamma_{\rm cusp}(a) \log(t/m^2) - \tilde{\mathcal{G}}_0(a)$ [Korchemsky; ...]

- dual conformal symmetry implies:
  $$M(p_i, m_i) = M(u, v)\,, \qquad u = \frac{m_1 m_3}{s + (m_1 - m_3)^2}\,, \quad v = \frac{m_2 m_4}{s + (m_2 - m_4)^2}$$

equal mass case	two-mass case
$m_i = m$	$m_1 = m_3 = m$
	$m_2 = m_4 = M$
$u = \frac{m^2}{s}\,, \quad v = \frac{m^2}{t}$	$u = \frac{m^2}{s}\,, \quad v = \frac{M^2}{t}$

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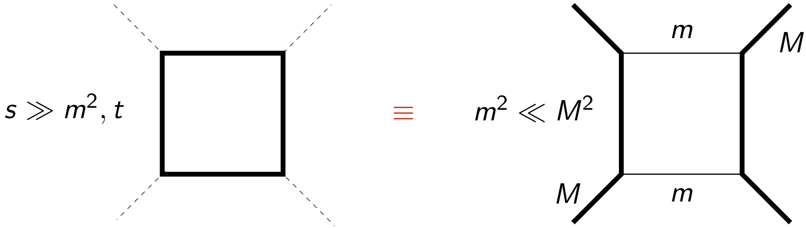
  Regge limit $s \gg m^2, t$ equivalent to $m^2 \ll M^2$ in "Bhabha-type" scattering



- determine leading Regge behavior of integrals [Eden et al, The analytic S-matrix]
- systematics of Regge limit simpler here compared to dimensional regularization

Regge limits for amplitudes on the Coulomb branch

- take Regge limit $s = (p_1 + p_2)^2 \to \infty$ [J. H., Naculich, Schnitzer, Spradlin, 2010]
  expect
  $$\beta(t/m^2)\left(\frac{s}{m^2}\right)^{\alpha(t/m^2)} + \mathcal{O}(m^2)$$
  trajectory $\alpha(t/m^2) = -\frac{1}{2}\Gamma_{\rm cusp}(a)\log(t/m^2) - \tilde{\mathcal{G}}_0(a)$ [Korchemsky; ...]
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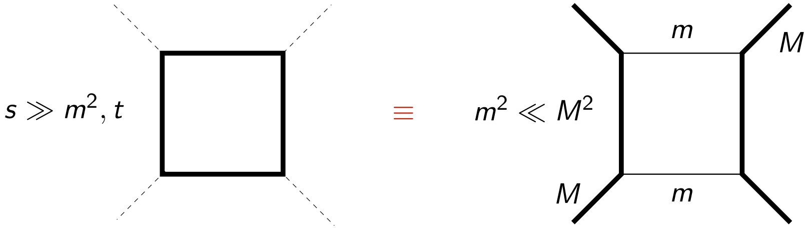

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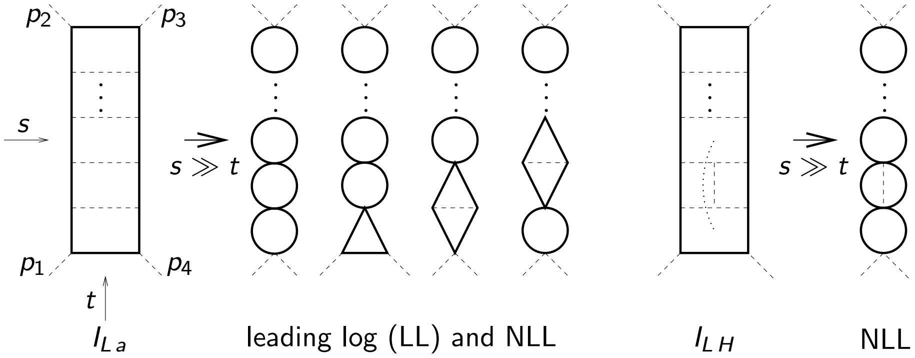
LL and NLL Regge limit to all loop orders



Regge limit to all loop orders:

- LL : ladder integrals
- NLL and LL : ladders and ladders with one H-shaped insertion

[J. H., Naculich, Schnitzer, Spradlin, 2010]

- in contrast, in dimensional regularization, many different diagrams contribute

Summary

- Higgs IR regulator for planar $\mathcal{N} = 4$ SYM
  - makes dual conformal symmetry exact
  - restricts integral basis
  - exponentiation of amplitude easier to compute
  - Regge limit: LL and NLL computed to all loop orders

Outlook

- advantages over dimensional regularization

  $\Rightarrow$ previously hard/impossible computations seem accessible

  e.g. two-loop amplitudes with $n \geq 6$ external particles

  e.g. can one compute the five-loop value of the cusp anomalous dimension?

- use analytic of amplitudes structure for finite m^2

- Yangian symmetry is essential part of

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  $\Rightarrow$ mass regulator relevant for extension to loop level

  momentum twistor variables might play an important role

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Extra slides

Exponentiation in Higgs regularization

- universal planar structure

$$\log M_4 = D(s) + D(t) + F_4(s/t) + \mathcal{O}(\epsilon)$$

- reminder: dimensional regularization $(\beta = 0)$

$$D(s) = -1/2 \sum a^\ell \left[\Gamma^{(\ell)}_{\text{cusp}}/(\ell\epsilon)^2 + \mathcal{G}^{(\ell)}_0/(\ell\epsilon)\right] \left(\mu^2/s\right)^{\ell\epsilon}$$

- finite part $F_4(s/t)$ is also simple! [Anastasiou, Bern, Dixon, Kosower 2002; Bern, Dixon, Smirnov, 2003]

$$F_4 = \frac{1}{2}\Gamma_{\text{cusp}}(a)\left[\frac{1}{2}\log^2\frac{s}{t} + \frac{2}{3}\pi^2\right] + c(a)$$

$M^{(2)} - \frac{1}{2}\left(M^{(1)}\right)^2$ interference $1/\epsilon \times O(\epsilon) = O(1)$

$\Rightarrow$ in order to compute $\log M$, need $O(\epsilon)$ terms in M

- analog in Higgs regularization [Alday, J. H., Plefka, Schuster, 2009; J. H., Naculich, Schnitzer, Spradlin, 2010]

[based on general field theory analysis by Korchemsky, Sterman, ...]

$$D(s) = -\frac{1}{4}\Gamma_{\text{cusp}}(a)\log^2\frac{s}{m^2} - \tilde{G}_0(a)\log\frac{s}{m^2}$$

F_4 equal up to scheme-dependent constant

we have $m^2 \times \log m^2 \to 0 \Rightarrow$ can drop all $O(m^2)$ terms

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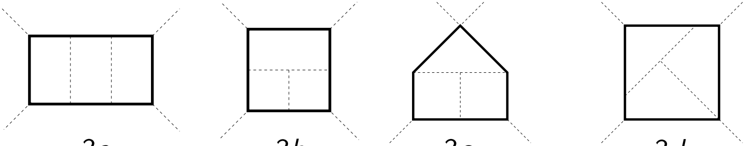
Three-loop exponentiation

For simplicity, set $s = t\,, \quad \log(\frac{m^2}{s}) \equiv L$ [J. H., Naculich, Schnitzer, Spradlin, 2010]

Infrared consistency:

$$M^{(3)} = -\frac{1}{6}L^6 + \frac{\pi^2}{12}L^4 + 2\zeta_3 L^3 + (-\frac{\pi^4}{30} - \frac{\Gamma^{(3)}_{\rm cusp}}{4})L^2 + \mathcal{O}(L)$$

On the other hand,



$$M_3 = 1/(-4)\,[c_{3a} I_{3a} + c_{3b} I_{3b} + c_{3c} I_{3c} + c_{3d} I_{3d}]$$

Compute...

$$I_{3a} = \frac{17}{90}L^6 + \frac{\pi^2}{9}L^4 + \ldots\,, \qquad I_{3c} = \mathcal{O}(L^0)\,,$$

$$I_{3b} = \frac{43}{180}L^6 - \frac{\pi^2}{9}L^4 + \ldots\,, \qquad I_{3d} = \mathcal{O}(L)$$

Hence $c_{3a} = 1\,, \quad c_{3b} = 2$

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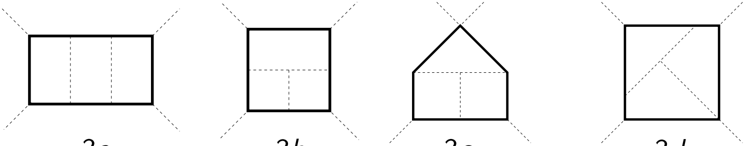

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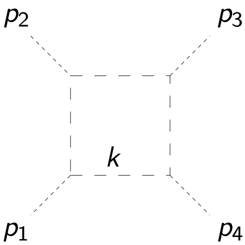
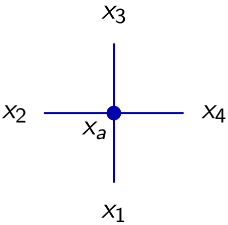
Dual conformal symmetry (1/2)

- observation: $\mathcal{N}=4$ SYM loop integrals have a dual conformal symmetry

[Drummond, J.H., Smirnov, Sokatchev, 2006]





- loop integrand has conformal symmetry in dual space

$$x^{\mu}_{i+1} - x^{\mu}_{i} = p_i$$

e.g. inversion symmetry $x^{\mu} \to x^{\mu}/x^2$ or special conformal transformations

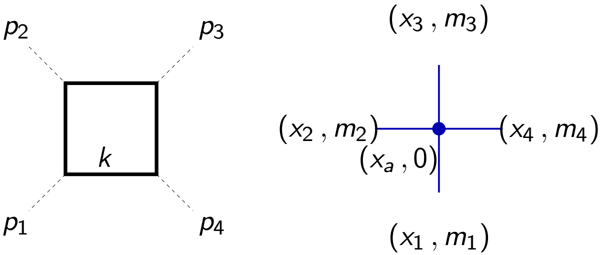
$$K^{\mu} = \sum_i \left[2x_i^{\mu}\, x_i^{\nu} \frac{\partial}{\partial x_{i\nu}} - x_i^2 \frac{\partial}{\partial x_{i\mu}} \right]$$

- breaking of symmetry $D = 4 - 2\epsilon$ under control [Drummond, J.H., Korchemsky, Sokatchev, 2007]

Dual conformal symmetry (2/2)

refinement: $U(N+M) \to U(N) \times U(1)^M$ [Alday, J.H., Plefka, Schuster, 2009]

- on-shell conditions now read $p_i^2 = (m_i - m_{+1})$
- particles in loop have mass m_i



- important: in addition to the dual coordinates x_i, we can vary the masses m_i

$$\hat{K}^\mu = K^\mu + \sum_i \left[2x_i^\mu\, m_i \frac{\partial}{\partial m_i} - m_i^2 \frac{\partial}{\partial x_{i\,\mu}} \right]$$

- integral has exact dual conformal symmetry

$$\hat{K}^\mu I = 0$$

- very natural from string theory: m corresponds to radial coordinate of AdS_5