Yangians and Grassmannians in $\mathcal{N} = 4$ super Yang-Mills

L. Ferro
LAPTH, Annecy-le-Vieux

based on work in collaboration with J. M. Drummond

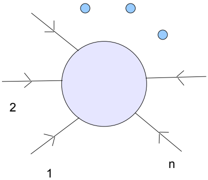
IGST 2010, Stockholm

Outline

1. Introduction
2. Yangian symmetry
3. Grassmannian formulæ
4. Conclusions

Motivations

$\mathcal{A}_n =$



Scattering amplitudes in $\mathcal{N} = 4$ SYM: why?

Remarkable features:

- share properties of QCD amplitudes but easier to compute
- constrained by hidden symmetries
- $\mathcal{N} = 4$ SYM dual to Type IIB on $AdS_5 \times S^5$ (AdS/CFT correspondence)
- planar limit: underlying integrable structure

Scattering amplitudes

On-shell supermultiplet of $\mathcal{N}=4$ super Yang-Mills theory: conveniently described by a superfield Φ

$$\Phi = G^{+} + \eta^{A}\Gamma_{A} + \frac{1}{2!}\eta^{A}\eta^{B}S_{AB} + \frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\overline{\Gamma}^{D} + \frac{1}{4!}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\epsilon_{ABCD}G^{-}$$

- η^{A}: Grassmann parameters
- $G^{+}, \Gamma_{A}, S_{AB}, \overline{\Gamma}^{A}, G^{-}$: positive helicity gluon, gluino, scalar, anti-gluino and negative helicity gluon states respectively
- helicity: $h\Phi = \Phi$
- $p^{2} = 0 \Longleftrightarrow p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}, \quad q^{\alpha A} = \lambda^{\alpha}\eta^{A}$

on-shell superspace: $(\lambda^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \eta^{A})$

Expansion in powers of Grassmann parameters η

$$\mathcal{A}(\Phi_1, \ldots, \Phi_n) = \mathcal{A}_n = \mathcal{A}_n^{\mathrm{MHV}} + \mathcal{A}_n^{\mathrm{NMHV}} + \ldots + \mathcal{A}_n^{\overline{\mathrm{MHV}}} = \mathcal{A}_{n;0}^{\mathrm{MHV}} \mathcal{P}_n$$

where

$$\mathcal{P}_n = \mathcal{P}_n^{\mathrm{MHV}} + \mathcal{P}_n^{\mathrm{NMHV}} + \ldots + \mathcal{P}_n^{\overline{\mathrm{MHV}}}$$

Helicity (homogeneity) condition

$$h_i \mathcal{A}(\Phi_1, \ldots, \Phi_n) = \mathcal{A}(\Phi_1, \ldots, \Phi_n), \qquad \mathrm{i} = 1, \ldots \mathrm{n}$$

and

$$h_i \mathcal{P}_n = 0, \quad \mathrm{i} = 1, \ldots \mathrm{n}$$

At tree-level

$$\mathcal{A}_{n;0} = \frac{\delta^4(p)\delta^8(q)}{\langle 12 \rangle \ldots \langle n1 \rangle} \mathcal{P}_{n;0}(\lambda_i, \tilde{\lambda}_i, \eta_i) = \mathcal{A}_{n;0}^{\mathrm{MHV}} \mathcal{P}_{n;0}, \quad \langle \mathrm{ij} \rangle = \lambda_i^\alpha \lambda_{j\alpha}$$

Scattering amplitudes

Beyond tree-level, $\mathcal{P}_n$ is IR divergent

$$\mathcal{P}_n = \sum_i c_i \mathcal{I}_i$$

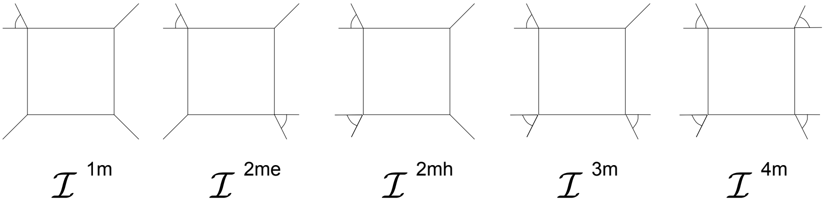
- $\mathcal{I}_i$: integral functions which contain IR divergences
- c_i: algebraic functions
- i: different integral topologies

Scattering amplitudes

One-loop: linear combination of scalar integral basis and algebraic functions

$$\mathcal{A}_{n;1} = \sum \left(a\mathcal{I}^{1\text{m}} + b\mathcal{I}^{2\text{me}} + c\mathcal{I}^{2\text{mh}} + d\mathcal{I}^{3\text{m}} + f\mathcal{I}^{4\text{m}} \right)$$

nm: number n of vertices in the box with more than 1 gluon



Symmetries: superconformal

Superconformal symmetry: expected [Witten]

- At tree-level (ignoring collinear anomalies)

$$j_a \mathcal{A}_n = 0$$

for any generator j_a of the superconformal algebra $psu(2,2|4)$

$$j_a \in \{p^{\alpha\dot{\alpha}}, q^{\alpha A}, \bar{q}^{\dot{\alpha}}_A, m_{\alpha\beta}, \bar{m}_{\dot{\alpha}\dot{\beta}}, r^A{}_B, d, s^\alpha_A, \bar{s}^A_{\dot{\alpha}}, k_{\alpha\dot{\alpha}}\}$$

- Beyond tree level: broken by IR divergences

Symmetries: dual superconformal

Hidden symmetry

Dual superconformal symmetry [Drummond, Henn, Korchemsky, Sokatchev]

- Conformal symmetry in the dual coordinate space:

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}, \qquad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^{\alpha}\eta_i^{A}$$

- Not related to ordinary superconformal symmetry
- Valid for planar limit

Strong coupling: dual superconformal symmetry arises naturally combining bosonic and fermionic T-duality

[Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf]

Symmetries: dual superconformal

Hidden symmetry

Dual superconformal symmetry[Drummond, Henn, Korchemsky, Sokatchev]

- Tree-level amplitudes are covariant under dual superconformal transformations

$$K^{\alpha\dot{\alpha}}\mathcal{A}_n = -\sum_i x_i^{\alpha\dot{\alpha}}\mathcal{A}_n$$

- Reformulate operators to have invariance

$$J'_a\mathcal{A}_n = 0$$

for any generator J'_a of the *dual* copy of $psu(2,2|4)$

$$J'_a \in \{P_{\alpha\dot{\alpha}}, Q_{\alpha A}, \bar{Q}^A_{\dot{\alpha}}, M_{\alpha\beta}, \overline{M}_{\dot{\alpha}\dot{\beta}}, R^A{}_B, D', S'^A_\alpha, \overline{S}^{\dot{\alpha}}_A, K'^{\alpha\dot{\alpha}}\}$$

- Broken beyond tree level

Symmetries

Superconformal and dual superconformal symmetries superpose

$$
\begin{array}{ccccccc}
 & p & & & & K & \\
q & & \bar{q} & = & \bar{S} & & S \\
s & & \bar{s} & = & \bar{Q} & & Q \\
 & k & & & & P &
\end{array}
$$

Superconformal + dual superconformal algebras:[Drummond, Henn, Plefka]

Yangian symmetry

Yangian symmetry

- Lie algebra: level-zero generators j_a

$$[j_a, j_b] = f_{ab}{}^c j_c$$

- Introduce level-one generators $j_a^{(1)}$

$$[j_a, j_b^{(1)}] = f_{ab}{}^c j_c^{(1)}$$

- Higher commutators constrained by the Serre relation

$$[j_a^{(1)}, [j_b^{(1)}, j_c]] + cyc(a,b,c) = h^2 (-1)^{|r||m|+|t||n|} \{j_l, j_m, j_n\} f_{ar}{}^l f_{bs}{}^m f_{ct}{}^n f^{rst}$$

The Yangian of the Lie algebra is generated by j and $j^{(1)}$

Yangian symmetry

- Level-zero generators: superconformal symmetry j_a
  Represented by a sum over single particle generators,

$$j_a = \sum_{k=1}^{n} j_{ka}$$

- Level-one generators: dual superconformal symmetry J'_a
  Bilocal formula [Dolan, Nappi, Witten]

$$j_a^{(1)} = f_a{}^{cb} \sum_{i<j} j_{ib} j_{jc}$$

Full symmetry of the tree-level amplitudes

$$y \mathcal{A}_n = 0$$

for any $y \in Y(psu(2,2|4))$

T-dual representation

- Level-zero: dual superconformal generators J_a [Drummond, L.F.]
  Symmetries of the function $\mathcal{P}_n$

$$J_a \mathcal{P}_n = 0$$

- Level-one: reformulated superconformal generators j'_a

$$k'_{\alpha\dot{\alpha}} \mathcal{P}_n = 0$$

where

$$k'_{\alpha\dot{\alpha}} = \sum_{i=1}^{n-1} \Big[\Big(\frac{\lambda_{i-1\ \alpha}}{\langle i-1\ i \rangle} - \frac{\lambda_{i+1\ \alpha}}{\langle i\ i+1 \rangle} \Big) \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} + \frac{\partial^2}{\partial \lambda_i^{\alpha} \partial \tilde{\lambda}_i^{\dot{\alpha}}} \Big]$$

Show that

$$k'_{\alpha\dot{\alpha}} \equiv J_a^{(1)} = f_a{}^{cb} \sum_{i<j} J_{ib} J_{jc}$$

T-dual representation

Go to momentum twistor space [Hodges] $\mathcal{W}_i^{\mathcal{A}} = (\lambda_i^{\alpha}, \mu_i^{\dot\alpha}, \chi_i^{A})$

$$\mu_i^{\dot\alpha} = x_i^{\alpha\dot\alpha}\lambda_{i\alpha}, \qquad \chi_i^{A} = \theta_i^{\alpha A}\lambda_{i\alpha}$$

standard twistor space associated to Wilson loop

- helicity condition
- spinor properties
- terms proportional to level-zero generators neglected

$$k'_{\alpha\dot\alpha} \equiv P^{(1)}_{\alpha\dot\alpha} = \sum_{i<j}\left[M^{\gamma}_{i\alpha}P_{j\gamma\dot\alpha} + \overline{M}^{\dot\beta}_{i\dot\alpha}P_{j\alpha\dot\beta} - D_i P_{j\alpha\dot\alpha} + \overline{Q}^{C}_{\dot\alpha i}Q_{j\alpha C} - (i \leftrightarrow j)\right]$$

Yangian generators

Twistor representation $\mathcal{Z}^{\mathcal{A}} = (\tilde{\mu}^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \eta^{A})$

$$j^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}}$$

$$j^{(1)\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j} (-1)^{\mathcal{C}} \left[\mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{C}}} \mathcal{Z}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{Z}_j^{\mathcal{B}}} - (i,j) \right]$$

$$j\mathcal{A}_n = j^{(1)}\mathcal{A}_n = 0$$

Momentum twistor representation $\mathcal{W}_i^{\mathcal{A}} = (\lambda_i^{\alpha}, \mu_i^{\dot{\alpha}}, \chi_i^{A})$

$$J^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}$$

$$J^{(1)\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j} (-1)^{\mathcal{C}} \left[\mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} \mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}} - (i,j) \right]$$

$$J\mathcal{P}_n = J^{(1)}\mathcal{P}_n = 0$$

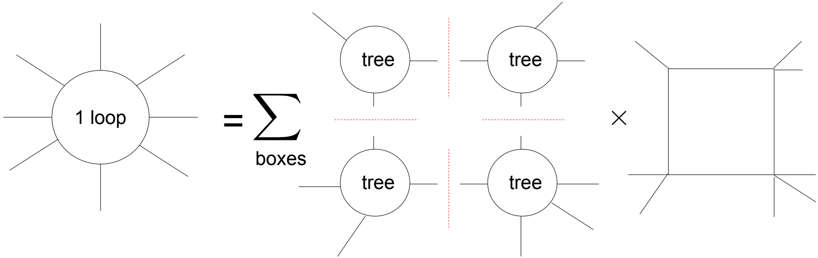
Leading singularities

At L-loops $\mathcal{A}$ is a $4L$-dimensional integral with branch cuts where internal propagator momenta go on-shell: examine discontinuities across them

The same discontinuities can have branch cuts: continue the process up to the leading singularity and perform all $4L$ integrals

Leading singularities

Any one-loop amplitude can be written as a linear combination of a basis of scalar integrals (IR div) with algebraic coefficients



Algebraic coefficients $\equiv$ Sum over leading singularities

Grassmannian formulæ

In twistor space

$$\mathcal{L}_{\mathrm{ACCK}} = \int \frac{\prod_{a,i} dc_{ai}}{\mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_n} \prod_{a=1}^{k} \delta^{4|4}\left(\sum_{i=1}^{n} c_{ai} \mathcal{Z}_i\right)$$

[Arkani-Hamed, Cachazo, Cheung, Kaplan]

Different integration contours give different leading singularities of $\mathrm{N}^{\mathrm{k}-2}\mathrm{MHV}$ amplitudes

- manifestly superconformal invariant
- c_{ai}: complex parameters which form a $(k \times n)$ matrix
- $\mathcal{M}_p = (p \ldots p + k - 1)$: determinant of $(k \times k)$ submatrix of the c_{ai}'s
- $GL(k)$ gauge symmetry

Grassmannian formulæ

Fix the gauge freedom by fixing k vectors

$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ \vdots & & & \vdots \\ c_{k1} & c_{k2} & \dots & c_{kn} \end{pmatrix} \to \left(\begin{array}{c|ccc} & c_{1k+1} & \dots & c_{1n} \\ \mathbb{1}_{k\times k} & \vdots & & \vdots \\ & c_{kk+1} & \dots & c_{kn} \end{array} \right)$$

$$\mathcal{L}_{\mathrm{ACCK}}(\mathcal{Z}) = \int \frac{d^{k\times(n-k)}c_{ai}}{\mathcal{M}_1 \dots \mathcal{M}_n|_{gf}} \prod_{a=1}^{k} \delta^{4|4}\left(\mathcal{Z}_a + \sum_{i=k+1}^{n} c_{ai}\mathcal{Z}_i\right)$$

Grassmannian formulæ

On-shell superspace: $(\lambda, \tilde{\lambda}, \eta)$

$$\mathcal{L}_{\text{ACCK}}(\lambda,\tilde{\lambda},\eta) = \int \frac{d^{k\times(n-k)}c_{ai}}{\mathcal{M}_1\ldots\mathcal{M}_n|_{gf}} \prod_{i=k+1}^{n} \delta^2\Big(\lambda_i - \sum_{a=1}^{k} c_{ai}\lambda_a\Big)$$
$$\times \prod_{a=1}^{k} \delta^2\Big(\tilde{\lambda}_a + \sum_{i=k+1}^{n} c_{ai}\tilde{\lambda}_i\Big)\delta^4\Big(\eta_a + \sum_{i=k+1}^{n} c_{ai}\eta_i\Big)$$

- c_{ai} : $k \times (n-k)$
- bosonic δ-fcs : $2(n-k) + 2k - 4$

To get info on amplitudes: $(k-2)\times(n-k-2)$ free c's variables

⬇

residua

Grassmannian formulæ

Example: $k = 2, \quad n = 4 \Longrightarrow$ MHV amplitude, no free variables
Example: $k = 3, n = 6 \Longrightarrow$ NMHV amplitude, 1 free variable τ

- Solve bosonic δ-fcs: identify $c_{ai}(\tau)$ explicitly

$$\mathcal{L} \propto \delta^4(p) \int \frac{d\tau}{[\mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_6](\tau)} \prod_{a=1}^{3} \delta^4\Big(\eta_a + \sum_{i=4}^{6} c_{ai}(\tau)\eta_i\Big)$$

- Each of the minors is linear in τ: 6 poles
- Leading singularities are linear combinations of residua

Grassmannian formulæ

In momentum twistor space

$$\mathcal{L}_{\text{MS}} = \int \frac{\prod_{a,i} dt_{ai}}{\mathcal{M}_1 \ldots \mathcal{M}_n} \prod_{a=1}^{k} \delta^{4|4}\left(\sum_{i=1}^{n} t_{ai} \mathcal{W}_i\right)$$ [Mason, Skinner]

Different integration contours give different leading singularities of $\mathrm{N^k MHV}$ amplitudes

- manifestly dual superconformal invariant
- MHV amplitude factored out, contributions to $\mathcal{P}_n$

Grassmannian formulæ

$$\mathcal{L}_{\mathrm{ACCK}}(\mathcal{Z})$$

superconformal invariant

↓

change of variables [Arkani-Hamed, Cachazo, Cheung]

↓

$$\mathcal{L}_{\mathrm{MS}}(\mathcal{W})$$

dual superconformal invariant

↓

Yangian invariance

Invariants of $Y(sl(m|m))$

- Consider the $sl(m|m)$ generators:

$$J^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}} = \sum_i \left(\begin{array}{c|c} W_i^{A'} \frac{\partial}{\partial W_i^{B'}} & W_i^{A'} \frac{\partial}{\partial \chi_i^{B}} \\ \hline \chi_i^{A} \frac{\partial}{\partial W_i^{B'}} & \chi_i^{A} \frac{\partial}{\partial \chi_i^{B}} \end{array} \right)$$

- invariants come with given Grassmann degree km
- Fourier transform bosonic variables: $W^{A'} \rightarrow \tilde{W}_{A'}$
- generators $Q^{A}_{A'} = \sum_i \chi_i^A \, \tilde{W}_{iA'}$
- degree zero: need all $\tilde{W}_i = 0 \rightarrow I_0 = \prod_i \delta^m(\tilde{W}_i) \rightarrow 1$
- degree m: need all $\tilde{W}_i$ proportional to $\tilde{W}_l$

$$I_m = \int dt \; f(t) \prod_{i \neq l} \delta^m(\tilde{W}_i - t_{li}\tilde{W}_l)\delta^m(\textstyle\sum_i t_{li}\chi_i)$$
$$\downarrow$$
$$\int dt \; f(t) \delta^{m|m}\left(\textstyle\sum_i t_{li}\mathcal{W}_i\right)$$

Invariants of $Y(sl(m|m))$

Generally we have

$$I_k = \int dt\, f(t) \prod_a \delta^{m|m}\Big(\sum_i t_{ai} \mathcal{W}_i\Big)\,, \qquad k \leq m$$

- similarly for $k \geq n - m$
- assume it for all k, though we have not proven it for $m < k < n - m$ [Korchemsky, Sokatchev]
- the measure is fixed by level-one generators $J^{(1)\mathcal{A}}{}_{\mathcal{B}}$

Yangian invariance

Direct proof of invariance [Drummond, L. F.]

Equivalence of the two versions: momentum twistors

$$J^{(1)\mathcal{A}}{}_{\mathcal{B}} \mathcal{L}_{MS}(\mathcal{W})$$

$$\int \frac{\prod_{a,m} dt_{am}}{\mathcal{M}_1 \dots \mathcal{M}_n} \left[\sum_{i<j} \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}} \mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} - \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}} \right] \prod_{a=1}^{k} \delta_a$$

- $\delta_a = \delta^{4|4}\left(\sum_{m=1}^{n} t_{am} \mathcal{W}_m\right)$
- not gauge-fixed
- neglect terms proportional to level-zero generators
- $\mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}}$ acts as a $gl(n)$ transformation on $\mathcal{W}_i$; acting on the delta functions we can replace

$$\mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} \to \mathcal{O}_{ij} \equiv \sum_{a=1}^{k} t_{ai} \frac{\partial}{\partial t_{aj}}$$

Yangian invariance

$$\sum_b \int \frac{\prod_{a,m} dt_{am}}{\mathcal{M}_1 \mathcal{M}_2 \dots \mathcal{M}_n} [\mathcal{O}_b^{\mathcal{A}} - \mathcal{V}_b^{\mathcal{A}}] (\partial_{\mathcal{B}} \delta_b) \prod_{a \neq b} \delta_a$$

where

$$\mathcal{O}_b^{\mathcal{A}} = \sum_{i<j} \mathcal{W}_i^{\mathcal{A}} \mathcal{O}_{ij} t_{bj}, \quad \mathcal{V}_b^{\mathcal{A}} = \sum_{i<j} \mathcal{W}_i^{\mathcal{A}} t_{bi}$$

- Commute $\mathcal{O}_b^{\mathcal{A}}$ back past the minors

$$\left[\frac{1}{\mathcal{M}_1 \dots \mathcal{M}_n}, \mathcal{O}_b^{\mathcal{A}} \right] = \frac{\mathcal{V}_b^{\mathcal{A}}}{\mathcal{M}_1 \dots \mathcal{M}_n}$$

- Remains a total derivative which can be neglected

$$\sum_b \sum_{i<j} \int \prod_{a,m} dt_{am} \mathcal{O}_{ij} \left[\mathcal{W}_i^{\mathcal{A}} t_{bj} \frac{1}{\mathcal{M}_1 \mathcal{M}_2 \dots \mathcal{M}_n} (\partial_{\mathcal{B}} \delta_b) \prod_{a \neq b} \delta_a \right]$$

Gauge-fixed case: new operator acting on the gauge-fixed part of the matrix

$$\mathcal{U}_{ij} = \sum_{r=k+1}^{n} t_{jr} \frac{\partial}{\partial t_{ir}}, \qquad 1 \le i < j \le k$$

$$\mathcal{O}_{ij} = \sum_{a=1}^{k} t_{ai} \frac{\partial}{\partial t_{aj}}, \qquad j \ge k$$

- $\mathcal{U}_{ij}$ acts as a $gl(k)$ rotation on the rows of non-gauge-fixed part
- Define $\mathcal{N}_b = (-\mathcal{U}_b, \mathcal{O}_b)$ and commute back past the minors: total derivative

Uniqueness of the invariant form

Is it possible to modify the form preserving invariance?

[Drummond, L. F.], [Korchemsky, Sokatchev]

$$\tilde{\mathcal{L}}_{n,k} = \int \frac{\prod_{a,m} dt_{am}}{\mathcal{M}_1 \dots \mathcal{M}_n} f(t) \prod_a \delta^{4|4}\Big(\sum_{m=1}^{n} t_{am}\mathcal{W}_m\Big)$$

- homogeneity condition

$$h_i \tilde{\mathcal{L}}_{n,k} = \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{A}}} \tilde{\mathcal{L}}_{n,k} = 0 \Longrightarrow \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{A}}} f(t) = 0$$

- preserve the fact that the variation of the integrand is a total derivative

$$J^{(1)\mathcal{A}}{}_{\mathcal{B}} \tilde{\mathcal{L}}_{MS}(\mathcal{W}) \to [\mathcal{N}_b^{\mathcal{A}}, f(t)] = 0$$

Uniqueness of the invariant form

Is it possible to modify the form preserving invariance?

[Drummond, L. F.], [Korchemsky, Sokatchev]

$$[\mathcal{N}_b^{\mathcal{A}}, f(t)] = 0$$

Is there any solution for $f(t)$ other than a constant?

- Look at the coefficients of the independent $\mathcal{W}_l$'s

$$N_{bl} = \sum_{a,j} O_{bl,aj} \frac{\partial}{\partial t_{aj}}$$

$$O_{bl,aj} = [\delta(j > l) - \delta(a \geq b)] t_{al} t_{bj}$$

- Linear dependence is given by the determinant of the matrix $O_{bl,aj}$

Uniqueness of the invariant form

Properties of the Yangian operator $J^{(1)\mathcal{A}}{}_{\mathcal{B}}$ + induction argument

$$\det O = [\mathcal{M}_1 \dots \mathcal{M}_n]^2$$

- f(t) must be constant almost everywhere
- f(t) can have discontinuities across the hyperplanes defined by the vanishing of the minors $\mathcal{M}_p$

Remark: relaxing homogeneity, f(t) can be different from a constant

Invariants of non-zero homogeneities

Relaxing the condition

$$h_i \tilde{\mathcal{L}}_{n,k} = \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{A}}} \tilde{\mathcal{L}}_{n,k} = 0$$

$$J^{(1)\mathcal{A}}{}_{\mathcal{B}} = \left[\sum_{i<j} \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}} \mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} - \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}} \right] + \frac{1}{2} \sum_i h_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}$$

one finds other invariants, as the determinant

$$\det \tilde{O} = \tilde{c}(k, n-k) [\mathcal{M}_1 \ldots \mathcal{M}_n]^2$$

can now vanish: $\tilde{c}(k, n-k) = 0$ for certain $k, n-k$.

- For instance $k = 1, n = 6$:

$$\sum_{a,j} \tilde{O}_{bl,aj} \frac{\partial}{\partial t_{aj}} \left(\frac{t_{11} t_{13} t_{15}}{t_{12} t_{14} t_{16}} \right) = 0$$

Conclusions

- Either standard or dual superconformal symmetry can be thought as the level-zero generators
- The two versions are related by T-duality
- Grassmannian formulæ: Yangian invariant and unique
- Could symmetry fix the full amplitude?
- Contribution of the holomorphic anomaly?