The classical *R*-matrix of AdS/CFT

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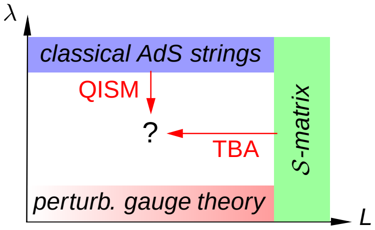
The pursuit of finiteness



TBA approach: Assumes integrability at finite λ, L.

- At $L \gg 1$, factorizability of the $\mathcal{S}$-matrix $\rightsquigarrow$ Fix 2-body $\mathcal{S}$-matrix using Yangian symmetry (universal $\mathcal{R}$-matrix?)
- Zamolodchikov's TBA trick $\rightsquigarrow$ Ground state energy $E_0(L)$.
  Claim: Excited states described by solutions of Y-system (boundary & analyticity conditions?).
  [Bombardelli-Tateo-Fioravanti, Frolov-Arutyunov, Gromov-Kazakov-Kozak-Vieira '09]

Need to prove integrability $\forall(\lambda, L)$ $\rightsquigarrow$ QISM.

Quantum Inverse Scattering Method

Starting point for QISM:

$$R_{\underline{12}}(u,v)\boldsymbol{L}^n_{\underline{1}}(u)\boldsymbol{L}^n_{\underline{2}}(v) = \boldsymbol{L}^n_{\underline{2}}(v)\boldsymbol{L}^n_{\underline{1}}(u)R_{\underline{12}}(u,v),$$

$$\boldsymbol{L}^n_{\underline{1}}(u)\boldsymbol{L}^m_{\underline{2}}(v) = \boldsymbol{L}^m_{\underline{2}}(v)\boldsymbol{L}^n_{\underline{1}}(u), \quad \forall n \neq m.$$

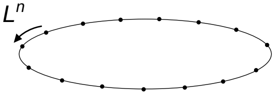
Defining monodromy $\boldsymbol{M}_{\underline{1}}(u) := \boldsymbol{L}^N_{\underline{1}}(u)\dots\boldsymbol{L}^1_{\underline{1}}(u)$, we have

$$\boldsymbol{T}(u) := \mathrm{tr}_{\underline{1}}\boldsymbol{M}_{\underline{1}}(u), \qquad \boxed{[\boldsymbol{T}(u),\boldsymbol{T}(v)] = 0, \quad \forall u,v.}$$

Classical limit and ClSM: Letting $R_{\underline{12}} = 1 + \hbar r_{\underline{12}} + O(\hbar^2)$ and $\boldsymbol{L}^n_{\underline{1}} = L^n_{\underline{1}} + O(\hbar)$ we find $\boxed{\{L^n_{\underline{1}}, L^m_{\underline{2}}\} = [r_{\underline{12}}, L^n_{\underline{1}}L^n_{\underline{2}}]\delta^{nm}.}$

Continuum limit:

$$L^n = P\overleftarrow{\exp}\int_{\sigma_n}^{\sigma_{n+1}} d\sigma L(\sigma)$$



yields $\boxed{\{L_{\underline{1}}, L_{\underline{2}}\} = [r_{\underline{12}}, L_{\underline{1}} + L_{\underline{2}}]\delta_{\sigma\sigma'}}$ $\rightsquigarrow$ Lie bialgebra structure.

Integrable Hamiltonian systems

Consider n-dim Hamiltonian system: $(\mathcal{P}; \{\cdot,\cdot\}), h \in C(\mathcal{P})$.

Definition

$\mu \in C(\mathcal{P})$ *is an integral of motion if*

- $\{\mu, h\} = 0$,
- $d\mu \neq 0$.

Definition (Integrable system)

$(\mathcal{P}, \{\cdot,\cdot\}, h)$ *is integrable if* $\exists \mu_1, \ldots, \mu_n \in C(\mathcal{P})$ *s.t.*

- $\{\mu_i, h\} = 0, \quad i = 1, \ldots, n$,
- $d\mu_1 \wedge \ldots \wedge d\mu_n \neq 0$,
- $\{\mu_i, \mu_j\} = 0, \quad i,j = 1, \ldots, n$.

Tasks for proving integrability:

(*i*) Identify the integrals of motion μ_i.

(*ii*) Show their involution $\{\mu_i, \mu_j\} = 0$.

Lax pair

Idea:[Lax] Obtain integrals μ_i from eigenvalues of a matrix L.

$\rightsquigarrow$ Reduces task (i) to spectral theory.

Suppose we can find $L \in Mat_{N\times N}[C(\mathcal{P})]$ whose evolution is 'isospectral', namely

$$\dot{L} := \{L, h\} = [M, L],$$

where $M \in Mat_{N\times N}[C(\mathcal{P})]$. Then also $\{L^j, h\} = [M, L^j]$, and

$$\{\mathrm{tr}\, L^j, h\} = 0, \qquad \forall j \in \mathbb{N}.$$

Hence spectrum of L provides integrals of motion of h.

So problem is reduced to finding such an L.

Dual of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra. Dual $\mathfrak{g}^*$ is a Poisson manifold:

$$\text{Lie bracket on } \mathfrak{g} \quad \rightsquigarrow \quad \text{Poisson bracket on } \mathfrak{g}^*.$$

Kostant-Kirillov bracket (KK-bracket):
Let $f \in C(\mathfrak{g}^*)$ and $L \in \mathfrak{g}^*$, then $(df)_L \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$. So define

$$\{f, g\}(L) := \langle L, [(df)_L, (dg)_L] \rangle.$$

Lax equation: $X \in \mathfrak{g} \simeq (\mathfrak{g}^*)^*$ defines $\hat{X} : L \mapsto \langle L, X \rangle$. Then

$$\frac{d}{dt}\langle L, X \rangle := \{\hat{X}, h\}(L) = \langle L, [X, (dh)_L] \rangle = \langle ad^*(dh)_L \cdot L, X \rangle.$$

Identifying $\mathfrak{g}^* \simeq \mathfrak{g}$ we have $ad^* \simeq ad$, and hence

$$\boxed{\dot{L} = ad^*(dh)_L \cdot L} \quad \Leftrightarrow \quad \dot{L} = [M, L], \;\; M := (dh)_L.$$

Setbacks

Any $h \in C(\mathfrak{g}^*)$ generates a Lax equation! Unlikely to provide a framework for describing non-trivial integrable systems.

Moreover,

Proposition

Spectral invariant functions $f \in C(\mathfrak{g}^*)$, i.e. ad^*-invariant functions, are Casimirs of the Kostant-Kirillov bracket.

In other words, the natural candidates $\operatorname{tr} L^j$ for integrable Hamiltonians all generate trivial flows under KK-bracket.

Resolution: Introduce a second Poisson bracket on $\mathfrak{g}^*$.

- Spectral invariants still characterised by KK-bracket, but
- Flows will be generated w.r.t. a different R-bracket.

Dual of a Lie *di*-algebra

Let $\mathfrak{g}$ be a Lie algebra. Given $R \in \mathrm{End}\,\mathfrak{g}$, introduce

$$[X, Y]_R := \tfrac{1}{2}\big([RX, Y] + [X, RY]\big).$$

Anti-symmetry of $[\cdot,\cdot]_R$ follows from that of $[\cdot,\cdot]$.
Sufficient condition for $[\cdot,\cdot]_R$ to satisfy Jacobi identity is

$$[RX, RY] - R\big([RX, Y] + [X, RY]\big) = -[X, Y], \quad \forall X, Y \in \mathfrak{g}.$$

This is the modified classical Yang-Baxter equation (mCYBE) and its solutions are classical R-matrices.

$(\mathfrak{g}, [\cdot,\cdot], [\cdot,\cdot]_R)$ is a Lie dialgebra. Its dual $\mathfrak{g}^*$ has two PBs

$$\{f, g\}(L) := \langle L, [(df)_L, (dg)_L]\rangle,$$
$$\{f, g\}_R(L) := \langle L, [(df)_L, (dg)_L]_R\rangle.$$

Constructing integrable systems

Theorem (Semenov-Tian-Shansky)

(i) Casimirs of KK-bracket are in involution w.r.t. R-bracket.

(ii) The flow generated by a Casimir h via R-bracket reads

$$\dot{L} = ad^* M \cdot L, \quad M := R(dh)_L.$$

This is the generalised Lax equation. When $\mathfrak{g}^* \simeq \mathfrak{g}$, it takes the form of the standard Lax equation.

Upshot: Allows construction of integrable systems on the dual $(\mathfrak{g}^*, \{\cdot,\cdot\}_R)$ of a Lie dialgebra $(\mathfrak{g}, R)$.

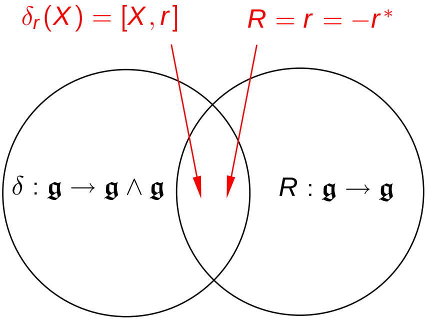
Lax matrix: Describing a specific model with phase-space $(\mathcal{P}, \{\cdot,\cdot\})$ requires a Poisson map

$$L : (\mathcal{P}, \{\cdot,\cdot\}) \to (\mathfrak{g}^*, \{\cdot,\cdot\}_R),$$

i.e. $\{L^* f, L^* g\} = L^* \{f, g\}_R$ for any $f, g \in C(\mathfrak{g}^*)$.

Lie bialgebras vs Lie dialgebras

Lie bialgebras
$(\mathfrak{g}, \delta)$



Lie dialgebras
$(\mathfrak{g}, R)$

Lie dialgebra: $R \in \mathrm{End}\,\mathfrak{g}$ defines a second bracket on $\mathfrak{g}$,

$$[x, y]_R = \tfrac{1}{2}\big([Rx, y] + [x, Ry]\big).$$

Lie bialgebra: $\delta^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ defines bracket on $\mathfrak{g}^*$.
In coboundary case $\delta_r^* : (\xi, \xi') \mapsto [\xi, \xi']_* = \tfrac{1}{2}\big([r\xi, \xi'] - [\xi, r^*\xi']\big)$.

Classical Yang-Baxter equation

Only restriction on the R-matrix is that it satisfies the mCYBE

$$[RX, RY] - R\big([RX, Y] + [X, RY]\big) = -[X, Y], \quad \forall X, Y \in \mathfrak{g}.$$

In tensor notation this reads

$$\boxed{[R_{\underline{12}}, R_{\underline{13}}] + [R_{\underline{12}}, R_{\underline{23}}] + [R_{\underline{32}}, R_{\underline{13}}] = -\hat{\omega}_{\underline{123}},}$$

where $\hat{\omega}(X, Y, Z) := \langle [X, Y], Z \rangle$.

For the r-matrix of a Lie bialgebra, $r + r^*$ is ad-invariant.
Imposing this further condition on R-matrix we obtain

$$[R_{\underline{12}}, R_{\underline{13}}] + [R_{\underline{12}}, R_{\underline{23}}] + [R_{\underline{13}}, R_{\underline{23}}] = -\hat{\omega}_{\underline{123}},$$

which is the usual mCYBE for $R = r$.

Zero curvature equation

The generalised Lax equation applies to any Lie (di)algebra $\mathfrak{g}$,

$$\dot{L} = ad^* M \cdot L.$$

So far we've used it to discuss only the Lax equation $\dot{L} = [M, L]$.

By choosing $\mathfrak{g}$ appropriately it is possible to cover also the zero-curvature equation

$$\partial_\tau \mathfrak{L} - \partial_\sigma \mathfrak{M} = [\mathfrak{M}, \mathfrak{L}].$$

Indeed, just need to find $\mathfrak{g}$ such that

$$ad^* \mathfrak{M} \cdot \mathfrak{L} = [\mathfrak{M}, \mathfrak{L}] + \partial_\sigma \mathfrak{M}.$$

Given by central extension $\hat{\mathfrak{G}}$ of current algebra $C^\infty(S^1, \mathring{\mathfrak{g}})$.

Centrally extended current algebras

Let $\mathring{\mathfrak{g}}$ be a 'little' Lie algebra with inner product $(\cdot,\cdot)$.
Consider $\mathfrak{G} := C^\infty(S^1,\mathring{\mathfrak{g}})$ with non-deg., inv., bilinear product

$$((\mathfrak{X},\mathfrak{Y})) := \int_{S^1} d\sigma(\mathfrak{X}(\sigma),\mathfrak{Y}(\sigma)), \quad \mathfrak{X},\mathfrak{Y}\in\mathfrak{G}.$$

Central extension: defined by the 2-cocycle

$$\omega(\mathfrak{X},\mathfrak{Y}) := \int_{S^1} d\sigma(\mathfrak{X}(\sigma),\partial_\sigma\mathfrak{Y}(\sigma)), \quad \mathfrak{X},\mathfrak{Y}\in\mathfrak{G}.$$

As a vector space $\hat{\mathfrak{G}} \equiv \mathfrak{G}\oplus\mathbb{C}$, equipped with the Lie bracket

$$[(\mathfrak{X},a),(\mathfrak{Y},b)] := ([\mathfrak{X},\mathfrak{Y}],\omega(\mathfrak{X},\mathfrak{Y})).$$

Extend also the product as $(((\mathfrak{X},a),(\mathfrak{Y},b))) := ((\mathfrak{X},\mathfrak{Y})) + ab$.

Lemma

If $(\mathring{\mathfrak{g}},R)$ is a Lie dialgebra, then so is $(\hat{\mathfrak{G}},R)$, where

$$R\big(\mathfrak{X}(\sigma),c\big) := \big(R(\mathfrak{X}(\sigma)),c\big), \quad \textit{for}\ \ (\mathfrak{X}(\sigma),c)\in\hat{\mathfrak{G}}.$$

Coadjoint action

Define the coadjoint action of $\hat{\mathfrak{G}}$ on $\hat{\mathfrak{G}}^*$ as

$$\langle\!\langle ad^*(\mathfrak{M}, c) \cdot (\mathfrak{X}, a), (\mathfrak{Y}, b) \rangle\!\rangle := -\langle\!\langle (\mathfrak{X}, a), [(\mathfrak{M}, c), (\mathfrak{Y}, b)] \rangle\!\rangle.$$

R.h.s. is independent of c, so center of $\hat{\mathfrak{G}}$ acts trivially.
Coadjoint action of $\mathfrak{G}$ on $\hat{\mathfrak{G}}^*$ reads

$$ad^*\mathfrak{M} \cdot (\mathfrak{X}, a) = (ad^*\mathfrak{M} \cdot \mathfrak{X} + a\, \partial_\sigma \mathfrak{M}, 0).$$

Since $a \in \mathbb{C}$ is invariant, we restrict attention to

$$\hat{\mathfrak{G}}^*_1 := \mathfrak{G}^* \oplus \{1\} \subset \mathfrak{G}^* \oplus \mathbb{C}.$$

Coadjoint action of $\mathfrak{G}$ on $\hat{\mathfrak{G}}^*_1 \simeq \mathfrak{G}^* \simeq \mathfrak{G}$ is therefore

$$\boxed{ad^*\mathfrak{M} \cdot (\mathfrak{X}, 1) = ([\mathfrak{M}, \mathfrak{X}] + \partial_\sigma \mathfrak{M}, 0).}$$

Constructing 2-d integrable field theories

The dual $\hat{\mathfrak{G}}^*$ of the Lie dialgebra $(\hat{\mathfrak{G}}, [\cdot,\cdot], [\cdot,\cdot]_R)$ has two PBs:

1) Kostant-Kirillov bracket: $\{f, g\}(\mathfrak{L}) := (\!(\mathfrak{L}, [df, dg])\!)$.
2) R-bracket: $\boxed{\tfrac{1}{2}\{f, g\}_R(\mathfrak{L}) := (\!(\mathfrak{L}, [df, dg]_R)\!).}$

Theorem

(i) Casimirs of KK-bracket are in involution w.r.t. R-bracket.

(ii) The flow generated by a Casimir h via R-bracket takes the form of a zero-curvature equation,

$$\partial_\tau \mathfrak{L} - \partial_\sigma \mathfrak{M} = [\mathfrak{M}, \mathfrak{L}], \quad \mathfrak{M} := R(dh)_{\mathfrak{L}}.$$

The Poisson manifold $(\hat{\mathfrak{G}}^*, \{\cdot,\cdot\}_R)$ therefore provides a very general setting for describing 2-d integrable field theories.

Different models correspond to different coadjoint orbits in $\hat{\mathfrak{G}}^*$.

r/s-matrix formalism

Let us write out the bracket $\{f, g\}_R$ for linear functions,

$$f : (\mathfrak{L}, 1) \mapsto (\!(\mathfrak{L}, \mathfrak{X})\!), \quad g : (\mathfrak{L}, 1) \mapsto (\!(\mathfrak{L}, \mathfrak{Y})\!),$$

where $\mathfrak{X} := X \cdot \delta_{\sigma_1}$, $\mathfrak{Y} := Y \cdot \delta_{\sigma_2}$ and $X, Y \in \mathring{\mathfrak{g}}$.
We then have, in the standard tensor notation

$$\{\mathfrak{L}_{\underline{\mathbf{1}}}, \mathfrak{L}_{\underline{\mathbf{2}}}\}_R = [R_{\underline{\mathbf{12}}}, \mathfrak{L}_{\underline{\mathbf{1}}}]\delta_{\sigma_1\sigma_2} - [R^*_{\underline{\mathbf{12}}}, \mathfrak{L}_{\underline{\mathbf{2}}}]\delta_{\sigma_1\sigma_2} + (R_{\underline{\mathbf{12}}} + R^*_{\underline{\mathbf{12}}})\delta'_{\sigma_1\sigma_2}.$$

This is the standard r/s-matrix algebra if we identify

$$r := \tfrac{1}{2}(R - R^*), \quad s := -\tfrac{1}{2}(R + R^*).$$

ultralocal models ($s = 0$) $\rightsquigarrow$ described by Lie bialgebra.
non-ultralocal models ($s \neq 0$) $\rightsquigarrow$ described by Lie dialgebra.

$\mathbb{Z}_4$-graded Lie superalgebra

Consider σ-models on semi-symmetric spaces [Zarembo '10]:

 $\longrightarrow$ $\text{super}(AdS_n \times Y_{10-n}) \equiv G/H$

Ingredients: Let $\mathring{\mathfrak{g}} = \mathring{\mathfrak{g}}_{\bar{0}} \oplus \mathring{\mathfrak{g}}_{\bar{1}}$ be a Lie superalgebra.

$\mathbb{Z}_4$-grading: Given by automorphism $\Omega : \mathring{\mathfrak{g}} \to \mathring{\mathfrak{g}}$, *s.t.* $\Omega^4 = 1$.

- $\mathring{\mathfrak{g}} = \mathring{\mathfrak{g}}_0 \oplus \mathring{\mathfrak{g}}_1 \oplus \mathring{\mathfrak{g}}_2 \oplus \mathring{\mathfrak{g}}_3$, where $\Omega(\mathring{\mathfrak{g}}_n) = i^n \mathring{\mathfrak{g}}_n$.
- $[\mathring{\mathfrak{g}}_n, \mathring{\mathfrak{g}}_m\} \subset \mathring{\mathfrak{g}}_{(n+m) \bmod 4}$, $\qquad \mathring{\mathfrak{g}}_{2n} \subset \mathring{\mathfrak{g}}_{\bar{0}}$, $\mathring{\mathfrak{g}}_{2n+1} \subset \mathring{\mathfrak{g}}_{\bar{1}}$.

Inner product: Non-deg., inv., bilinear form $\langle \cdot, \cdot \rangle : \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathbb{C}$.

- $\langle \mathring{\mathfrak{g}}_n, \mathring{\mathfrak{g}}_m \rangle = 0$ unless $n + m = 0 \bmod 4$.

Grassmann envelope: $\mathfrak{g} = (\Gamma \otimes \mathring{\mathfrak{g}})_{\bar{0}}$ is an ordinary Lie algebra.

- $\mathfrak{g}$ inherits corresponding properties from $\mathring{\mathfrak{g}}$.
- $G := \exp \mathfrak{g}$ and $H := \exp \mathfrak{g}_0$ (using $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$).

$\mathbb{Z}_4$-graded supercoset σ-models

Want σ-model for maps $\Sigma = \mathbb{R} \times S^1 \to G/H$. So let

$$g : \Sigma \to G, \quad A = -g^{-1}dg \in \Omega^1(\Sigma, \mathfrak{g})$$

and impose

- Global left G-action: under $g \mapsto Ug$, $U \in G$, have $A \mapsto A$.
- Local right H-action: under $g \mapsto gh$, $h : \Sigma \to H$ have $A \mapsto h^{-1}Ah - h^{-1}dh$, hence

$$A^{(1,2,3)} \mapsto h^{-1}A^{(1,2,3)}h, \qquad \text{where} \quad A^{(n)} \in \mathfrak{g}_n.$$

Possible Lagrangians (matter part of GS and PS s-strings):

$$\mathcal{L}_{\mathrm{GS}} := -\tfrac{1}{2}\langle A^{(2)} \wedge *A^{(2)}\rangle - \tfrac{1}{2}\langle A^{(1)} \wedge A^{(3)}\rangle + \langle \Lambda, dA - A^2\rangle,$$

$$\mathcal{L}_{\mathrm{PS}} := -\tfrac{1}{2}\langle (A - A^{(0)}) \wedge *(A - A^{(0)})\rangle + \tfrac{1}{2}\langle A^{(1)} \wedge A^{(3)}\rangle + \langle \Lambda, dA - A^2\rangle.$$

Hamiltonian formalism

Phase-space $\mathcal{P}$ parametrised by $(A_1^{(0,1,2,3)}, \Pi_1^{(0,1,2,3)})$ with

$$\{A_{1\underline{1}}^{(i)}(\sigma), \Pi_{1\underline{2}}^{(4-i)}(\sigma')\}_{\text{P.B.}} = C_{\underline{12}}^{(i\,4-i)}\delta(\sigma-\sigma').$$

Constraints: $\{\Phi^A \approx 0\}$

GS: first class $\mathcal{T}_\pm \approx \mathcal{C}^{(0)} \approx 0$, (partly) second class $\mathcal{C}^{(1,3)} \approx 0$.

PS: first class $\hat{\mathcal{T}}_\pm := \mathcal{T}_\pm + \frac{1}{2}\langle \mathcal{C}^{(1)}, \mathcal{C}^{(3)}\rangle \approx \mathcal{C}^{(0)} \approx 0$.

Extended Hamiltonian: $\mathcal{H} = \sum_A \rho_A \Phi^A$

$$\mathcal{H}_{\text{GS}} = \underbrace{\rho_+\mathcal{T}_+ + \rho_-\mathcal{T}_-}_{\text{conformal tr.}} - \underbrace{\langle \mu^{(3)}, \mathcal{C}^{(1)}\rangle - \langle \mu^{(1)}, \mathcal{C}^{(3)}\rangle}_{\kappa-\text{symmetry}} - \underbrace{\langle \mu^{(0)}, \mathcal{C}^{(0)}\rangle}_{\text{coset}},$$

$$\mathcal{H}_{\text{PS}} = \underbrace{\hat{\rho}_+\hat{\mathcal{T}}_+ + \hat{\rho}_-\hat{\mathcal{T}}_-}_{\text{conformal tr.}} - \underbrace{\langle \hat{\mu}^{(0)}, \mathcal{C}^{(0)}\rangle}_{\text{coset}}.$$

Hamiltonian Lax matrix

Look for Lax matrix $\mathfrak{L}$ as a linear combination of $(A_1^{(i)}, \Pi_1^{(i)})$, *s.t.*

$$\{\mathfrak{L}, P_0\} = \partial_\sigma \mathfrak{M} + [\mathfrak{M}, \mathfrak{L}],$$

for some $\mathfrak{M}$, where P_0 is energy.

A careful Hamiltonian analysis of both GS and PS yields

$$\mathfrak{L}_{\text{GS}} = \mathfrak{L}_{\text{BPR}}(z) + \frac{1}{2\sqrt{\lambda}}(1 - z^4)\left(\mathcal{C}^{(0)} + z^{-3}\mathcal{C}^{(1)} + z^{-1}\mathcal{C}^{(3)}\right),$$

$$\mathfrak{L}_{\text{PS}} = \mathfrak{L}_{\text{p.s.}}(z)\Big|_{\text{ghosts}=0} + \frac{1}{2\sqrt{\lambda}}(1 - z^4)\,\mathcal{C}^{(0)}.$$

Surprisingly we find the same result $\mathfrak{L}_{\text{GS}} = \mathfrak{L}_{\text{PS}} =: \mathfrak{L}$. Explicitly,

$$\boxed{\mathfrak{L}(z) = \sum_{j=1}^{4} z^j A_1^{(j)} + \frac{1 - z^4}{4z^4}\left(\sum_{j=1}^{4} j\, z^j A_1^{(j)} + 2\sum_{j=1}^{4} z^j (\nabla_1 \Pi_1)^{(j)}\right).}$$

Twisted loop algebra

Consider loop algebra $\mathcal{L}\mathfrak{g} := \mathfrak{g}[\![z, z^{-1}]\!]$ with decomposition

$$\mathcal{L}\mathfrak{g} = \mathcal{L}\mathfrak{g}_+ \dot{+} \mathcal{L}\mathfrak{g}_-,$$

where

- $\mathcal{L}\mathfrak{g}_+ := \mathfrak{g}[\![z]\!]$ consists of formal Taylor series in z,
- $\mathcal{L}\mathfrak{g}_- := z^{-1}\mathfrak{g}[z^{-1}]$, polys in z^{-1} without const. term.

$\mathbb{Z}_4$-twist: Notice $\Omega(\mathfrak{L}(z)) = \mathfrak{L}(iz)$. So extend $\Omega : \mathfrak{g} \to \mathfrak{g}$ as

$$\hat{\Omega} : \mathcal{L}\mathfrak{g} \to \mathcal{L}\mathfrak{g}, \quad \hat{\Omega}(X)(z) = \Omega(X(-iz)).$$

The twisted loop algebra is $\mathcal{L}\mathfrak{g}^\Omega := \{X \in \mathcal{L}\mathfrak{g} \,|\, \hat{\Omega}(X) = X\}$.
In particular $\mathcal{L}\mathfrak{g}^\Omega = \mathcal{L}\mathfrak{g}^\Omega_+ \dot{+} \mathcal{L}\mathfrak{g}^\Omega_-$ and

$$\mathfrak{L} \in C^\infty(S^1, \mathcal{L}\mathfrak{g}^\Omega).$$

Twisted inner product

Lax matrix can be rewritten as

$$\boxed{\mathfrak{L} = 4\,\phi(z)^{-1} \sum_{k=1}^{\infty} z^k \left(k\,A_1^{(k)} + 2\,(\nabla_1 \Pi_1)^{(k)} \right),}$$

where
$$\phi(z) := \frac{16z^4}{(1-z^4)^2}.$$

Introduce a twist in the standard inner product on $\mathcal{L}\mathfrak{g}^{\Omega}$:

$$(X, Y)_{\phi} := \oint \frac{dz}{2\pi i z} \phi(z) \langle X(z), Y(z) \rangle = \oint \frac{du}{2\pi i} \langle X(z), Y(z) \rangle.$$

The Zhukovsky variable u plays a central role in AdS/CFT,

$$u = 2\frac{1+z^4}{1-z^4}.$$

Recall $\langle \mathfrak{g}_n \cdot z^n, \mathfrak{g}_m \cdot z^m \rangle = \langle \mathfrak{g}_n, \mathfrak{g}_m \rangle z^{n+m} = 0$ if $n+m \neq 0$ (mod 4).

Smooth dual

'little' Lie algebra:

$$\boxed{\mathring{\mathfrak{g}} := \mathcal{L}\mathfrak{g}^{\Omega} \text{ with bilinear product } (\cdot,\cdot)_{\phi}.}$$

Current algebra: $\mathfrak{G} = C^{\infty}(S^1, \mathring{\mathfrak{g}})$ inherits twisted inner product,

$$(\!(\mathfrak{X}, \mathfrak{Y})\!)_{\phi} := \int_{S^1} d\sigma (\mathfrak{X}(\sigma), \mathfrak{Y}(\sigma))_{\phi}.$$

Let $\mathfrak{G}_{\pm} := C^{\infty}(S^1, \mathring{\mathfrak{g}}_{\pm})$ and $\mathfrak{G}^{\perp}_{\pm} := C^{\infty}(S^1, \mathring{\mathfrak{g}}^{\perp}_{\pm})$ where

$$\mathring{\mathfrak{g}}_{+} = \oplus_{n \geq 0} \mathfrak{g}_{(n)} \cdot z^n, \qquad \mathring{\mathfrak{g}}_{-} = \oplus_{n < 0} \mathfrak{g}_{(n)} \cdot z^n,$$
$$\mathring{\mathfrak{g}}^{\perp}_{+} = \oplus_{n > 0} \mathfrak{g}_{(n)} \cdot z^n, \qquad \mathring{\mathfrak{g}}^{\perp}_{-} = \oplus_{n \leq 0} \mathfrak{g}_{(n)} \cdot z^n.$$

With respect to $(\!(\cdot,\cdot)\!)_{\phi}$ we have $\mathfrak{G}^{*}_{-} \simeq \phi^{-1} \mathfrak{G}^{\perp}_{+}$ and so

$$\boxed{\mathfrak{L} \in \mathfrak{G}^{*}_{-}.}$$

Standard R-matrix

With respect to the decomposition $\mathring{\mathfrak{g}} = \mathring{\mathfrak{g}}_+ \dotplus \mathring{\mathfrak{g}}_-$, let

$$R := \pi_+ - \pi_-,$$

where $\pi_\pm : \mathring{\mathfrak{g}} \to \mathring{\mathfrak{g}}_\pm$ are projections. It satisfies mCYBE,

$$[RX, RY] - R([RX, Y] + [X, RY]) = -[X, Y], \qquad \forall X, Y \in \mathring{\mathfrak{g}},$$

so that

$$[X, Y]_R := \tfrac{1}{2}([RX, Y] + [X, RY]),$$

defines a second Lie bracket on $\mathring{\mathfrak{g}} = \mathcal{L}\mathfrak{g}^\Omega$ (dialgebra).

Remark: Due to the twist in the inner product, R is not skew:

$$R^* = -\varphi^{-1} \circ R \circ \varphi, \quad \varphi(z) = \phi(z) z^{-1}.$$

r/s-matrices

Tensor kernels: Given $\mathcal{O} : \mathring{\mathfrak{g}} \to \mathring{\mathfrak{g}}$, define $\mathcal{O}_{\underline{\mathbf{12}}} \in \mathring{\mathfrak{g}} \otimes \mathring{\mathfrak{g}}$ by

$$(\mathcal{O}X)_{\underline{\mathbf{1}}} = (\mathcal{O}_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{2}}})_{\phi \underline{\mathbf{2}}}.$$

Projection kernels are

$$\pi_{\pm \underline{\mathbf{12}}} = \sum_{m=1}^{\infty} \left(\frac{z_1}{z_2} \right)^{\pm m} C_{\underline{\mathbf{12}}}^{(\pm m \; \mp m)} \phi(z_2)^{-1},$$

where $C_{\underline{\mathbf{12}}} = C_{\underline{\mathbf{12}}}^{(00)} + C_{\underline{\mathbf{12}}}^{(13)} + C_{\underline{\mathbf{12}}}^{(22)} + C_{\underline{\mathbf{12}}}^{(31)}$ is tensor Casimir.

Recall that $r = \frac{1}{2}(R - R^*)$ and $s = -\frac{1}{2}(R + R^*)$, or explicitly

$$r_{\underline{\mathbf{12}}} = \text{v.p.} \frac{1}{z_2^4 - z_1^4} \left[\sum_{j=0}^{3} z_1^{4-j} z_2^j C_{\underline{\mathbf{12}}}^{(4-j\, j)} \phi(z_1)^{-1} + \sum_{j=0}^{3} z_1^j z_2^{4-j} C_{\underline{\mathbf{12}}}^{(j\, 4-j)} \phi(z_2)^{-1} \right],$$
$$s_{\underline{\mathbf{12}}} = \frac{1}{z_2^4 - z_1^4} \left[\sum_{j=0}^{3} z_1^{4-j} z_2^j C_{\underline{\mathbf{12}}}^{(4-j\, j)} \phi(z_1)^{-1} - \sum_{j=0}^{3} z_1^j z_2^{4-j} C_{\underline{\mathbf{12}}}^{(j\, 4-j)} \phi(z_2)^{-1} \right].$$

These are exactly the r/s-matrices of superstring [Magro '08].

Conclusions & outlook

- Integrable structure of AdS/CFT at $\lambda \gg 1$ is given by a Lie dialgebra, with standard R-matrix but twisted inner product.
- Although loop algebra $\mathcal{L}\mathfrak{g}^{\Omega}$ is written in the z-variable, the Zhukovsky map $z \mapsto u$ enters naturally in inner product:

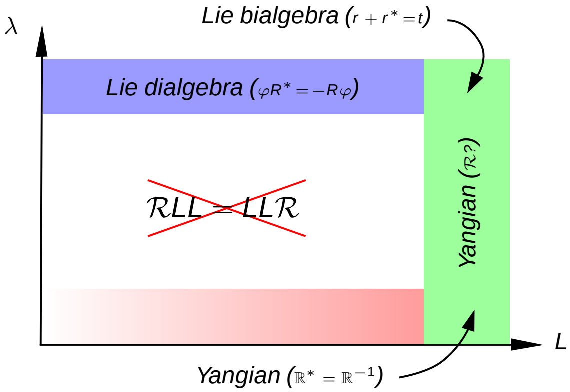
$$(X, Y)_{\phi} = \oint \frac{du}{2\pi i} \langle X(z), Y(z) \rangle,$$

  where $\langle X(z), Y(z) \rangle$ is a formal Laurent series in u.
- Other $\mathbb{Z}_m$-gradings are also known to give rise to actions admitting a Lax connection [Young '05]. In this case twist and Zhukovsky map should be

$$\phi(z) = \frac{4m\, z^m}{(1 - z^m)^2}, \qquad \boxed{u = 2\frac{1 + z^m}{1 - z^m}}$$

- Generalise to include ghosts and compare GS to PS?
- How to quantize Lie dialgebas?

Integrable structures in AdS/CFT



TACK SJÄLV