

# 2d inverse cascade: renormalization group theory versus numerical experiments

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# Introduction

- i Kolmogorov-Kraichnan's theory of 2-d turbulence

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- ii Renormalization group theory of 2-d turbulence

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- ii Renormalization group theory of 2-d turbulence
- iii Numerical experiments

# Velocity and Vorticity in 2d

## The Navier Stokes equation

$$(\partial_t + v \cdot \partial)v^\alpha - \nu \partial^2 v^\alpha = -\partial^\alpha P + f^\alpha, \quad \alpha = 1, 2$$

$$\partial \cdot v = 0$$

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in *two dimensions* transports the **vorticity** field

$$\omega := \epsilon_{\alpha\beta} \partial^\alpha v^\beta, \quad \omega = \text{vorticity}$$

$$\partial_t \omega + v \cdot \partial \omega - \nu \partial^2 \omega = \epsilon_{\alpha\beta} \partial^\alpha f^\beta$$

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in *two dimensions* transports the **vorticity** field

$$\begin{aligned}\omega &:= \epsilon_{\alpha\beta} \partial^\alpha v^\beta, & \omega &= \text{vorticity} \\ \partial_t \omega + v \cdot \partial \omega - \nu \partial^2 \omega &= \epsilon_{\alpha\beta} \partial^\alpha f^\beta\end{aligned}$$

Conservation of vorticity moments in the inviscid limit.



# Kraichnan's picture of 2d turbulence

# Energy and Enstrophy

Total Energy and Enstrophy equations

$$\partial_t \int d^2x v^2 = -2\nu \int d^2x \omega^2 \equiv -2\nu \Omega^2$$

$$\partial_t \int d^2x \omega^2 = -2\nu \int d^2x (\partial_\alpha \omega_\beta \partial^\alpha \omega^\beta)$$

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- Energy dissipation vanishes as  $\nu \downarrow 0$ .
- Energy conservation  $\implies$  energy transfer towards **small wavenumbers**.
- Enstrophy dissipation at **large wavenumbers**.

# Kàrmàn-Howarth-Monin equation

Gaussian, time short-correlated translational invariant forcing

$$\langle f^\alpha(x, t) f_\alpha(y, s) \rangle = \delta(t - s) F(x - y, m)$$

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the Kàrmàn-Howarth-Monin (KHM) equation (equal times) is

$$\begin{aligned} \frac{1}{2} \langle (\partial \cdot \delta v)(x) \delta v(x)^2 \rangle &= \\ &= \partial_t \langle v(x) \cdot v(0) \rangle - F(x, m) - 2\nu \langle (\partial_\alpha v)(x) (\partial^\alpha v)(0) \rangle \end{aligned}$$

# Hypotheses encoding Kraichnan's theory:

- i velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points:

$$\lim_{x \rightarrow 0} \partial_t \langle v(x) \cdot v(0) \rangle = I_{\mathcal{E}}$$

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- iii No **energy** dissipative anomalies occur:

$$\left\{ \lim_{\nu \downarrow 0} \lim_{x \downarrow 0} - \lim_{x \downarrow 0} \lim_{\nu \downarrow 0} \right\} \nu \langle \partial_\alpha v^\beta(x, t) \partial^\alpha v_\beta(0, t) \rangle = 0$$

# KHM equation and "mean field" scaling

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$$\frac{1}{2} \partial_\mu S_3^\mu = \partial_t \langle v(x) \cdot v(0) \rangle - F(x, m) - 2\nu D_2(x)$$

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## Inverse Energy Cascade

$$\langle \delta v_{\parallel}^3 \rangle = \langle \delta v_{\parallel} \delta v_{\perp}^2 \rangle \stackrel{mx \gg 1}{=} \frac{3 I_\varepsilon x}{2} \quad \text{mean field} \quad \Rightarrow \quad \delta v \sim x^{1/3}$$

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## Direct Enstrophy Cascade

$$\langle \delta v_{\parallel}^3 \rangle = \langle \delta v_{\parallel} \delta v_{\perp}^2 \rangle \stackrel{\ell x \ll mx \ll 1}{=} \frac{I_\Omega x^3}{8} \quad \text{mean field} \quad \Rightarrow \quad \delta v \sim x$$



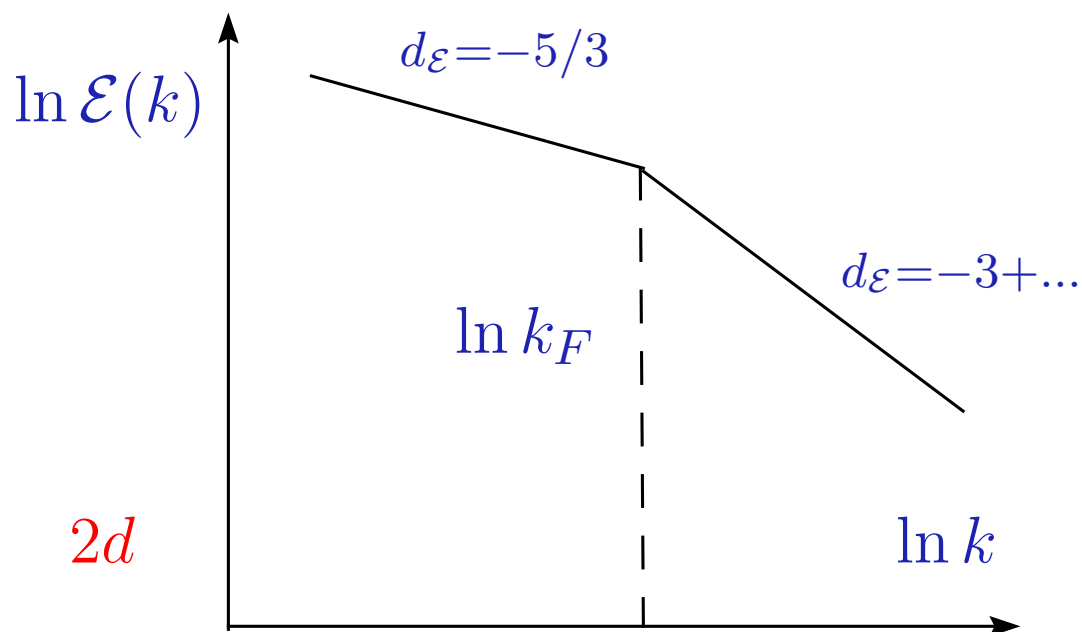
# Energy spectrum

$$\mathcal{E}(k) = \int \frac{d^d p}{(2\pi)^d} \delta(|k| - |p|) \int d^d x e^{ip \cdot x} \prec v(x) \cdot v(0) \succ$$

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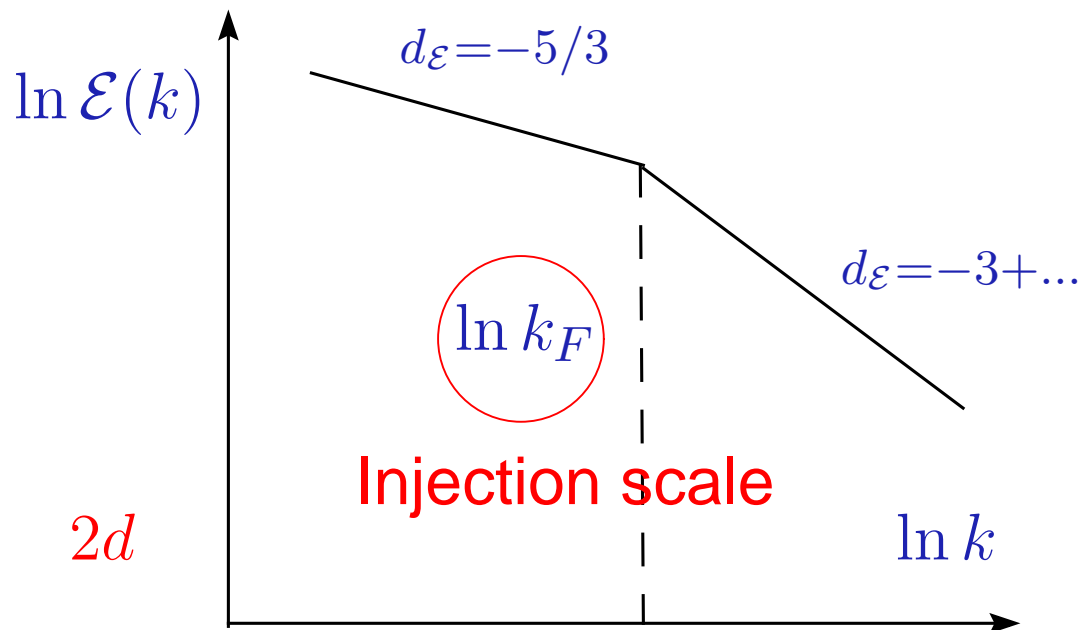
scaling arguments suggest ( $d = 2$ )



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# UV Renormalization Group analysis

**P. Olla (1991),(1994)**  
**Honkonen et al. (1998)**

# Power law forcing

$$(\partial_t + v \cdot \partial)v^\alpha - \nu \partial^2 v^\alpha = -\partial^\alpha P + f^\alpha - \frac{v^\alpha}{\tau} \quad \alpha = 1, 2$$

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Eckman friction

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with *power law spectrum*  $d = 2$

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$$\check{F}(p) = \frac{g_1 \nu^3 h_1(p, M, m)}{p^{d-4+2\varepsilon}} + g_2 \nu^3 p^2 h_2(p, M, m)$$

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- $\tau$  seems to play no role for RG !

# RG prediction

## Energy spectrum

$$\mathcal{E}(q) = \varepsilon^{1/3} g_1^{2/3} \nu^2 q^{1 - \frac{4\varepsilon}{3}} R \left[ \varepsilon, \frac{m}{q}, \left( \frac{q_b}{q} \right)^{2 - \frac{2\varepsilon}{3}} \right] \quad (\star)$$

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If  $R$  has a limit for  $q \downarrow 0$  the scaling prediction of  $(\star)$  coincides with the **scale by scale** balance prediction

$$d_v - d_t = -\frac{d_t}{2} - d_x (2 - \varepsilon)$$

$$2 d_v - d_x = -\frac{d_t}{2} - d_x (2 - \varepsilon)$$

# Numerics

# Model

We numerically integrate the *vorticity* equation

$$\omega = \epsilon_{\alpha\beta} \partial^\alpha v^\beta$$

$$\partial_t \omega + v \cdot \partial \omega = (-1)^{p+1} \nu \Delta^p \omega + \frac{(-1)^{q+1}}{\tau} \Delta^{-q} \omega + f_\omega$$

- Resolution up to  $2048^2$

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- $(-1)^{p+1} \nu \Delta^p \omega$ : hyperviscosity.

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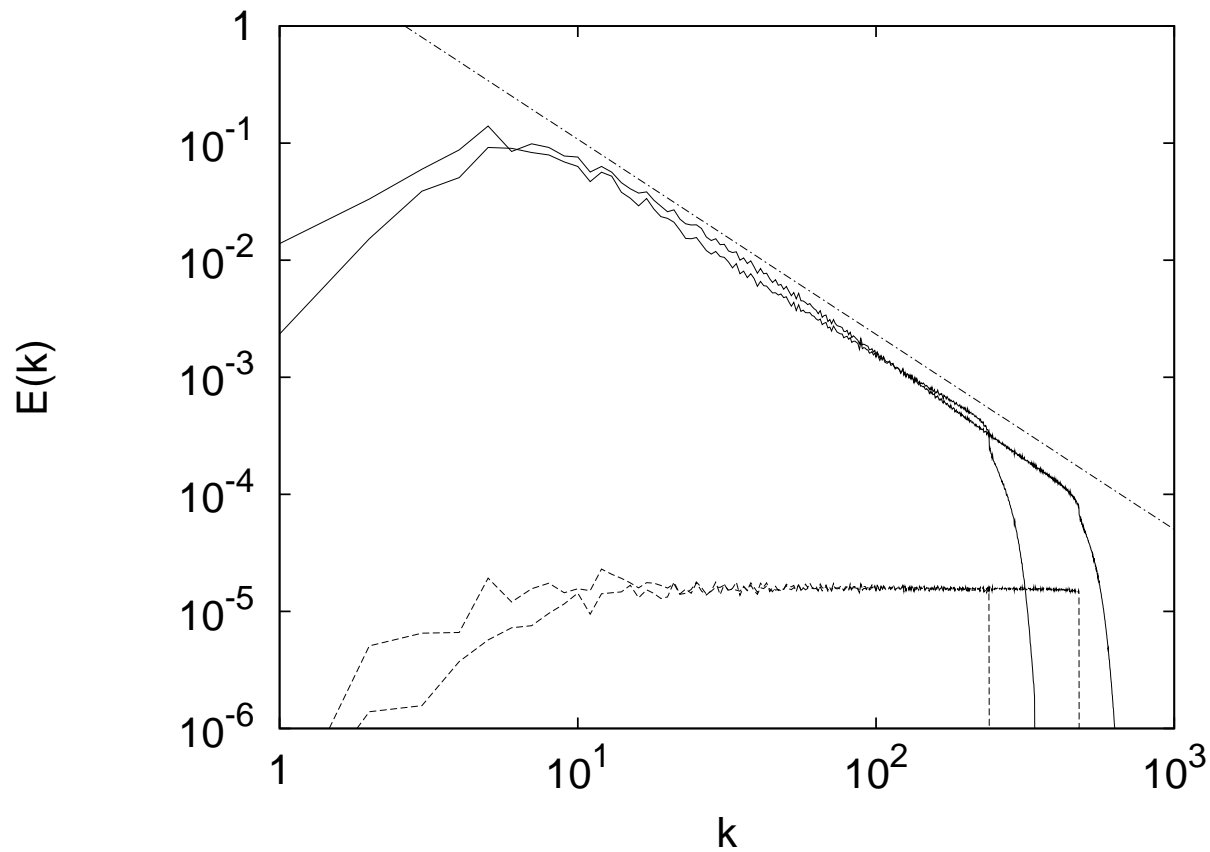
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- $(p, q) = (4, 0)$  for  $\varepsilon \leq 2$
- $(p, q) = (4, 1)$  for  $\varepsilon \geq 3$  to inquire the inverse cascade.

# Numerical Energy Spectra

“Small scale forcing”  $\varepsilon \leq 2$

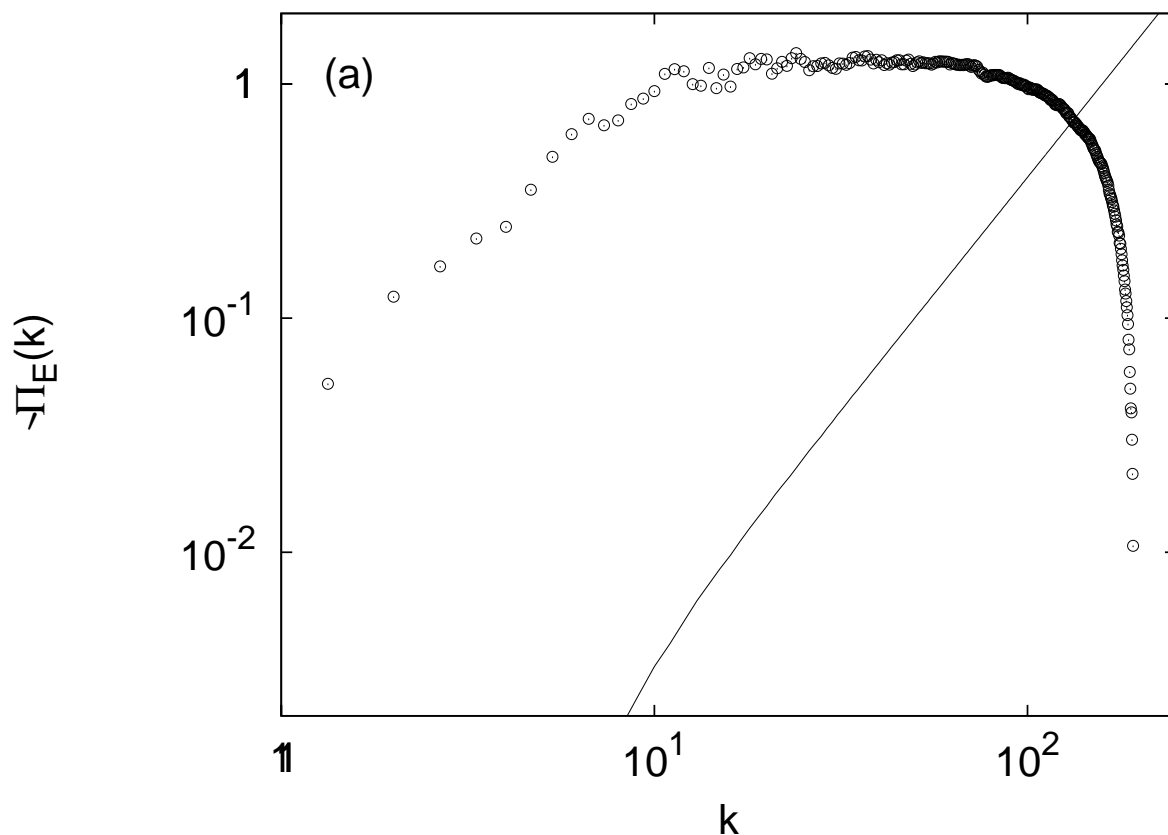




# Energy flux at $\varepsilon = 1$

*Energy flux* for  $\varepsilon = 1$  and integrated *energy input*

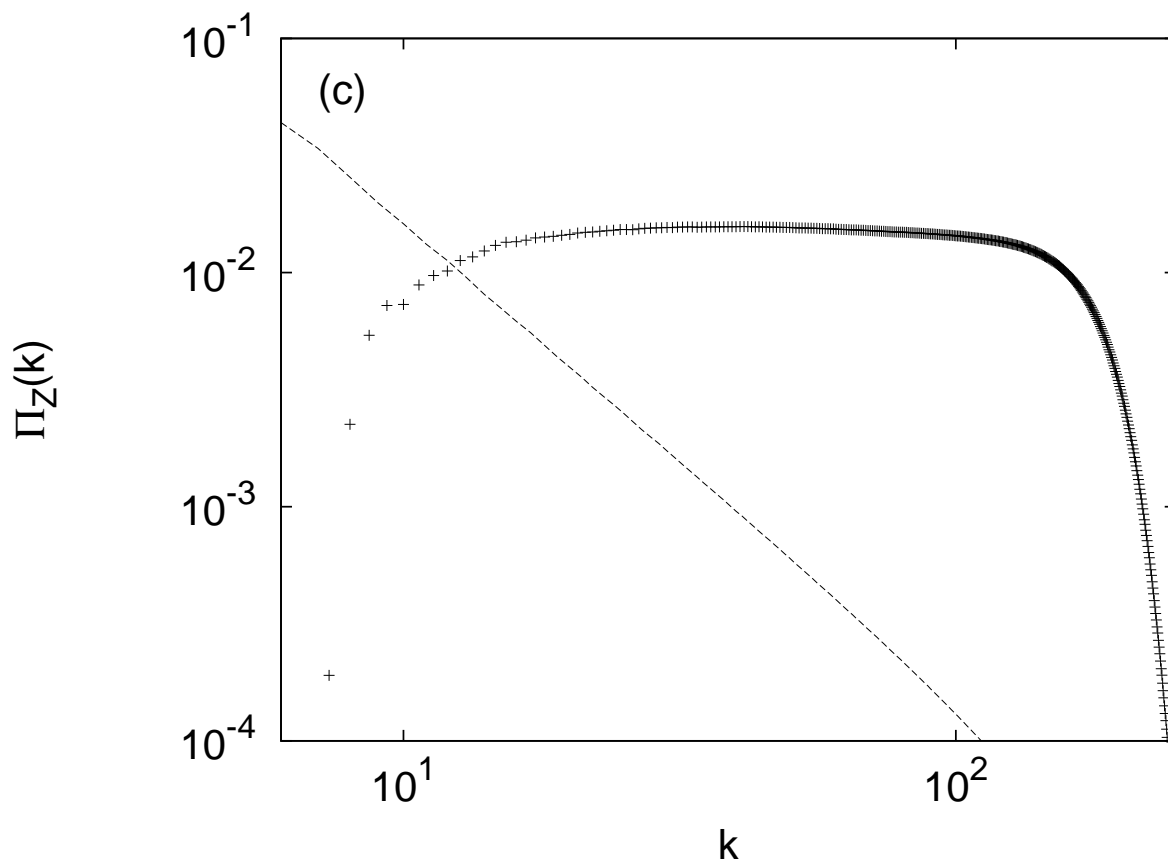
$$\int_{k \leq p} \frac{d^d k}{(2\pi)^d} \frac{1}{k^{d-4+2\varepsilon}} \Big|_{d=2, \varepsilon=1} \sim p^2$$



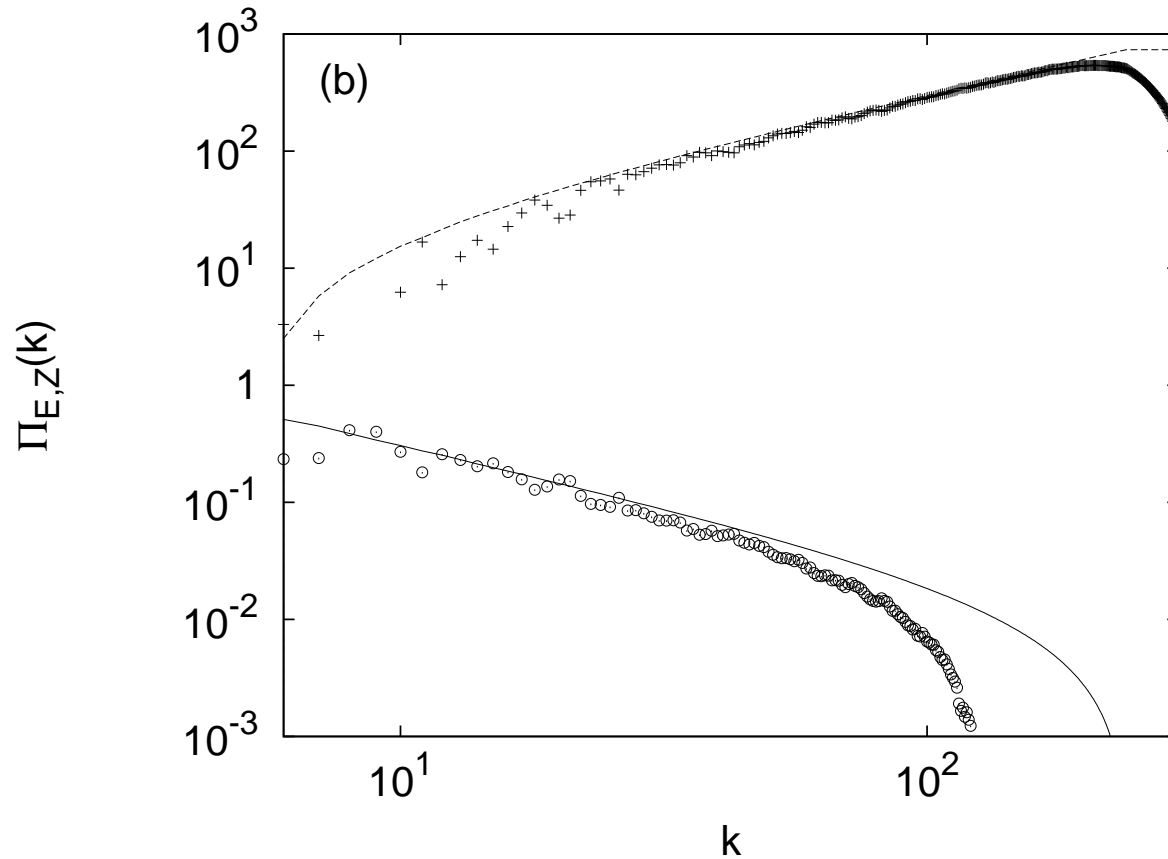
# Enstrophy flux at $\varepsilon = 4$

*Enstrophy flux* for  $\varepsilon = 4$  and integrated *enstrophy input*

$$\int_{k \geq p} \frac{d^d k}{(2\pi)^d} \frac{k^2}{k^{d-4+2\varepsilon}} \Big|_{d=2, \varepsilon=4} \sim p^{-2}$$



$$\varepsilon = 2.5$$



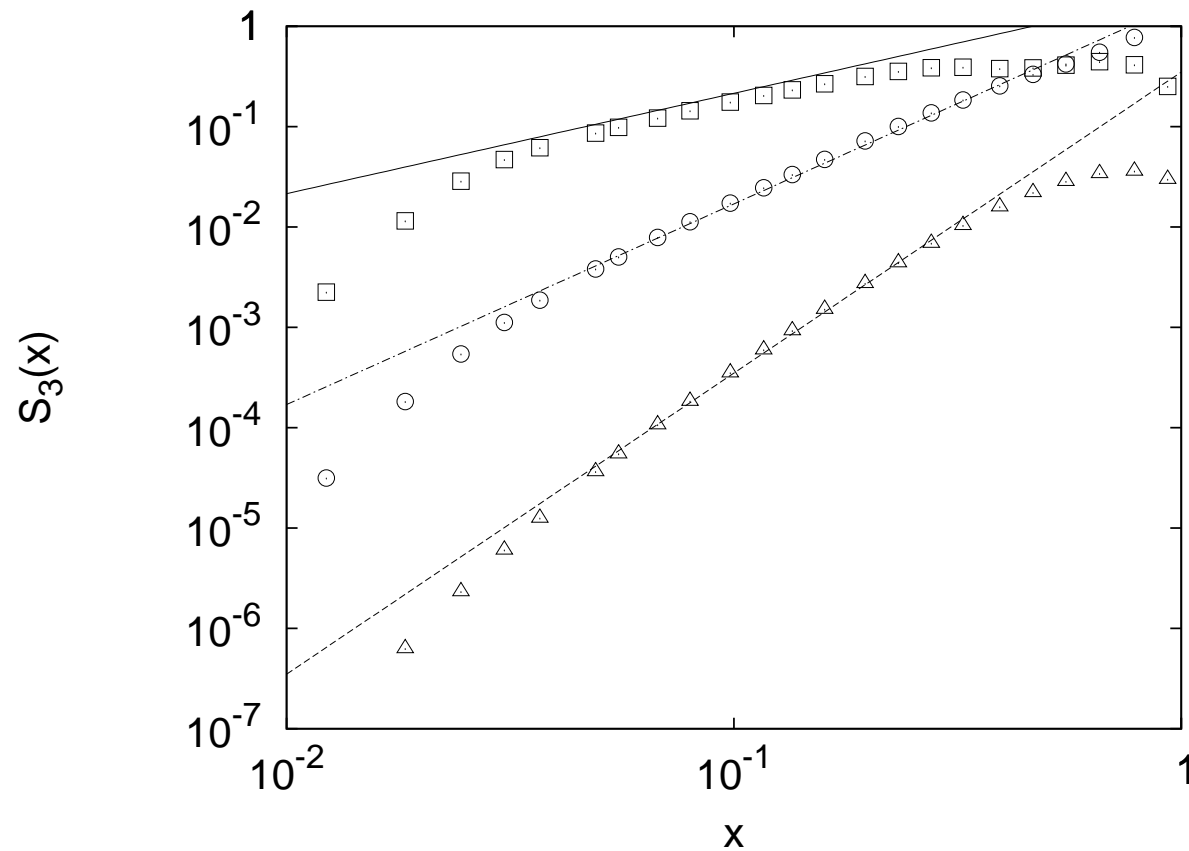
● circles:  $\mathcal{E}$ -flux

line: integrated  $\mathcal{E}$ -input

● crosses:  $\Omega$ -flux

dashes: integrated  $\Omega$ -input

# Third order velocity structure functions



- $\epsilon = 1$  (squares):  $\delta v \sim x^{1/3}$  (inverse cascade).
- $\epsilon = 2.5$  (circles):  $\delta v \sim x^{-1 + \frac{2\epsilon}{3}}$  (dimensional).
- $\epsilon = 4$  (triangles):  $\delta v \sim x$  (direct cascade).

# KHM for power law forcing: $0 \leq \varepsilon \leq 2$

$$-\frac{1}{2} \partial_\mu \prec \delta v^\mu(x) \delta \mathbf{v}^2(x) \succ =$$
$$\frac{\Sigma_2}{(2\pi)^2} \frac{F_0 A(-4 + 2\varepsilon, 2)}{(4 - 2\varepsilon) x^{4-2\varepsilon}} - \prec (\partial_t v_\beta)(x) v^\beta(0) \succ$$

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By hypotheses (i), (ii)

$$\prec (\partial_t v_\beta)(x) v^\beta(0) \succ = \frac{\Sigma_2}{(2\pi)^2} \frac{M^{4-2\varepsilon} \bar{F}_{4-2\varepsilon}}{4 - 2\varepsilon}$$

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The ratio between forcing and energy injection rate is

$$\frac{\partial_t \langle v_\beta(x) v^\beta(0) \rangle}{F^\alpha_\alpha(x)} \propto (Mx)^{4-2\varepsilon} \gg 1$$

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$$\langle \delta v_{\parallel}^3 \rangle \simeq M^{4-2\varepsilon} F_0 c'_1 x + \frac{c'_2 F_0}{x^{3-2\varepsilon}} + \dots$$



# KHM for power law forcing: $2 < \varepsilon < 3$

$$\partial_t \langle v_\beta(x) v^\beta(0) \rangle - \frac{1}{2} \partial_\mu \langle \delta v^\mu(x) \delta \mathbf{v}^2(x) \rangle \asymp \alpha$$

$$\frac{m^{4-2\varepsilon} \bar{F}_{4-2\varepsilon}}{(2\varepsilon-4)} + \frac{m^{4-2\varepsilon} (mx)^2 \bar{F}_{6-2\varepsilon}}{4(6-2\varepsilon)} - \frac{F_0 A(2\varepsilon-4, 2) x^{2\varepsilon-4}}{(2\varepsilon-4)} + \dots$$

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The injection rate is

$$\langle (\partial_t v_\beta)(x) v^\beta(0) \rangle = \frac{\Sigma_2}{(2\pi)^2} \frac{m^{4-2\varepsilon} \bar{F}_{4-2\varepsilon}}{4-2\varepsilon}$$

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The KHM reduces to

$$-\frac{1}{2} \partial_\mu \langle \delta v^\mu(x) \delta \mathbf{v}^2(x) \rangle =$$

$$\frac{\Sigma_2}{(2\pi)^2} \left\{ \frac{m^{4-2\varepsilon} (mx)^2 \bar{F}_{6-2\varepsilon}}{4(6-2\varepsilon)} - \frac{F_0 A(2\varepsilon-4, 2) x^{2\varepsilon-4}}{(2\varepsilon-4)} + \dots \right\}$$

# KHM for power law forcing: $2 < \varepsilon < 3$

$$\partial_t \langle v_\beta(x) v^\beta(0) \rangle - \frac{1}{2} \partial_\mu \langle \delta v^\mu(x) \delta \mathbf{v}^2(x) \rangle \propto \alpha$$

$$\frac{m^{4-2\varepsilon} \bar{F}_{4-2\varepsilon}}{(2\varepsilon-4)} + \frac{m^{4-2\varepsilon} (mx)^2 \bar{F}_{6-2\varepsilon}}{4(6-2\varepsilon)} - \frac{F_0 A(2\varepsilon-4, 2) x^{2\varepsilon-4}}{(2\varepsilon-4)} + \dots$$

The structure function scales as

$$\langle \delta v_{\parallel}^3 \rangle \stackrel{(mx) \ll 1}{\simeq} F_0 c'_2 x^{2\varepsilon-3} \{ 1 + c'_1 (mx)^{6-2\varepsilon} + \dots \}$$

# KHM for power law forcing: $3 < \varepsilon$

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In the infinite volume case

$$-\frac{1}{2} \partial_\mu \langle \delta v^\mu(x) \delta \mathbf{v}^2(x) \rangle = \frac{\Sigma_2}{(2\pi)^2} \left\{ \frac{m^{4-2\varepsilon} (m x)^2 \bar{F}_{6-2\varepsilon}}{4(6-2\varepsilon)} - \frac{F_0 A(2\varepsilon-4, 2) x^{2\varepsilon-4}}{(2\varepsilon-4)} + \dots \right\}$$

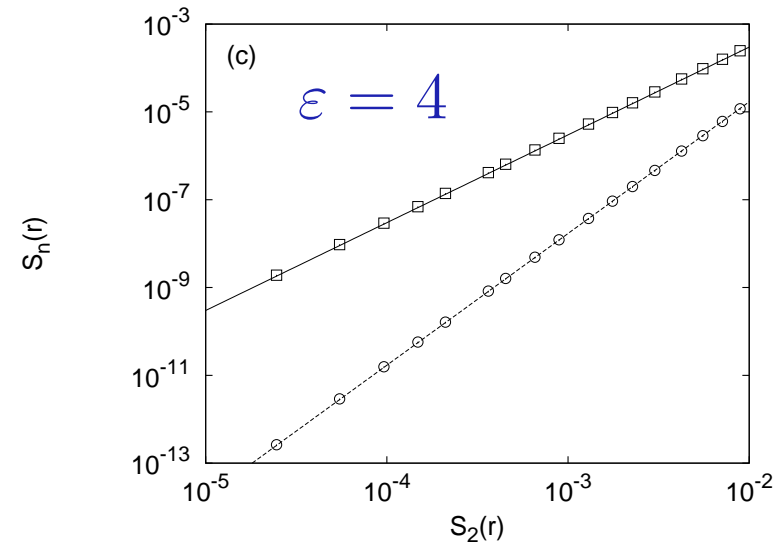
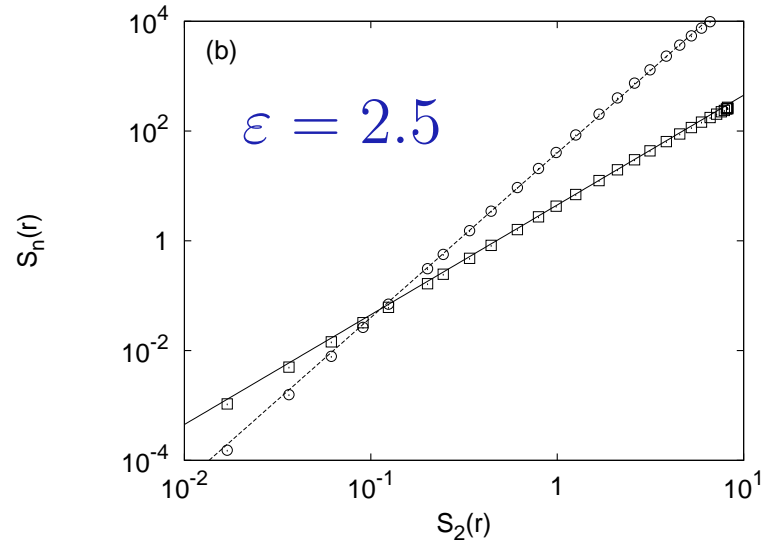
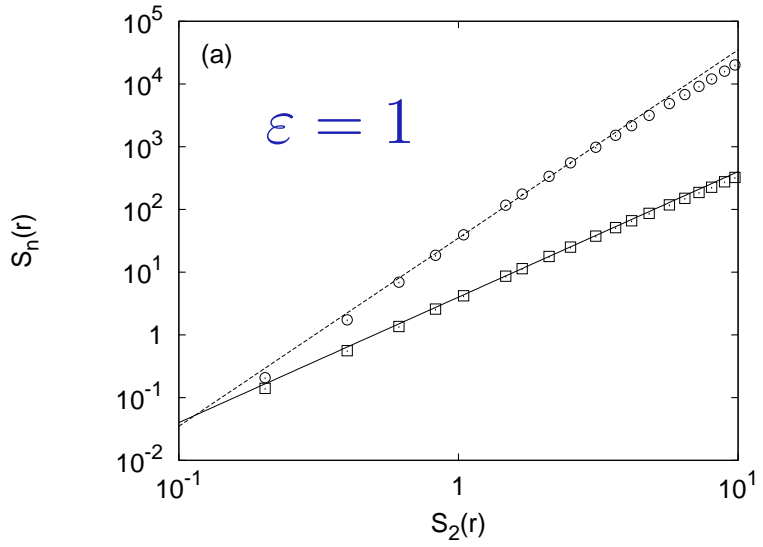
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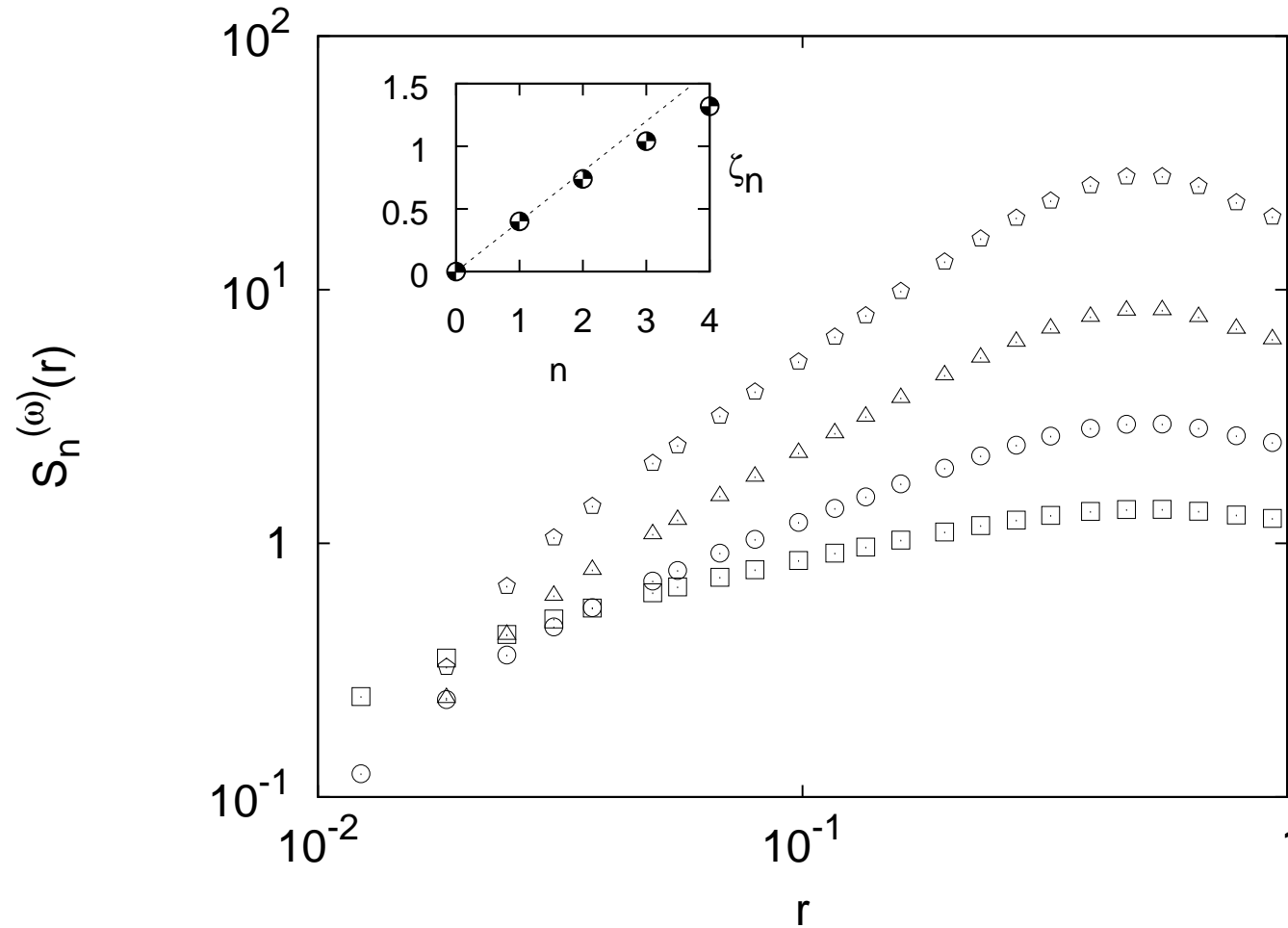
$$\langle \delta v_{\parallel}^3 \rangle \stackrel{(m x) \ll 1}{\simeq} m^{6-2\varepsilon} F_0 c'_1 x^3 \{ 1 + c'_2 (m x)^{2\varepsilon-6} + \dots \}$$

# No velocity struct. fun. intermittency





# vorticity intermittency (direct cascade)?



- $\varepsilon = 4$
- Structure functions of order  $n = 1, 2, 3, 4$

# Conclusions

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- why the discrepancy?