TBA and functional relations for the AdS_5/CFT_4 correspondence

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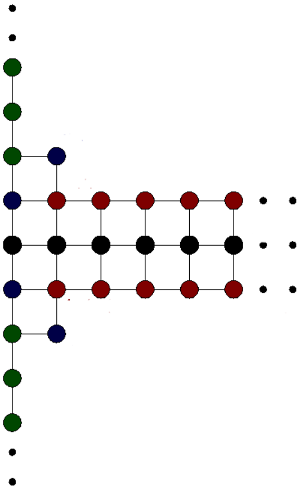
Nordita 2010

Gromov-Kazhakov-Viera:



$$
\begin{aligned}
Y_{a,b}(u+i/g)Y_{a,b}(u-i/g) &= (1+Y_{a+1,b}(u))(1+Y_{a-1,b}(u)) \\
&\times \left(1+\frac{1}{Y_{a,b+1}(u)}\right)^{-1}\left(1+\frac{1}{Y_{a,b-1}(u)}\right)^{-1}
\end{aligned}
$$

Outline

1. Integrability and the planar limit of $\mathcal{N} = 4$ SYM
2. The thermodynamic Bethe ansatz
3. The TBA for $\mathcal{N} = 4$ SYM
4. TBA $\equiv$ dispersion relation
5. Extended Y-system
6. Conclusions

In $\mathcal{N}=4$ SYM with $su(N_c)$ gauge group, consider a single trace operator of the form

$$O(x) = Trace[\phi_{i_1}(x)\phi_{i_2}(x)\ldots\phi_{i_L}(x)],$$

and calculate its anomalous dimension Δ from the two-point function

$$\langle O(x)O(y)\rangle \sim \frac{1}{|x-y|^{2\Delta}},$$

where

$$\Delta = \Delta_0 + \gamma(\lambda)\ , \quad \lambda = 4\pi^2 g^2 = N_c g_{YM}^2.$$

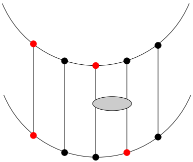
Planar limit of $\mathcal{N}=4$ SYM theory → Integrable structures!

Mixing matrix ↔ Integrable quantum spin-chain Hamiltonian

Example [Minahan-Zarembo]: one loop $su(2)$ sector:

$$O_1 = Trace[ZWWZW], \quad O_2 = Trace[ZWZWW],$$

$$\langle O_1(x) O_2(y) \rangle$$



→ xxx-spin chain!

$$\left(\frac{v_j + i/2}{v_j - i/2} \right)^L = - \prod_{k=1}^{M} \frac{v_j - v_k + i}{v_j - v_k - i},$$

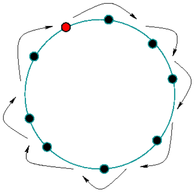
$$\gamma(\lambda) \sim \frac{i\lambda}{8\pi^2} \sum_{j=1}^{M} \left(\frac{1}{v_j + i/2} - \frac{1}{v_j - i/2} \right).$$

- [Beisert-Staudacher] → all loop asymptotic BA (ABA)
  In the $su(2)$ sector:

$$\left(\frac{x_d(u_j + i/g)}{x_d(u_j - i/g)}\right)^L = -\prod_{k=1}^{M} \frac{u_j - u_k + i2/g}{u_j - u_k - i2/g}(\sigma(u_j, u_k))^2,$$

($u_i = 2v_i/g$)

- Problem: the ABA equations are ok only up to order g^{2L} !
- ABA ∼ Bethe-Yang S-matrix quantisation conditions:



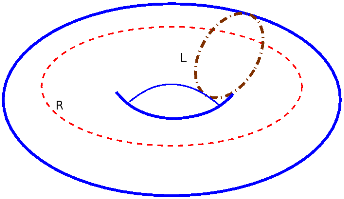
$$e^{iLp_j} \prod_{k \neq j} S(\theta_j, \theta_k) = 1 \;,\; \text{(ABA/Bethe-Yang)}.$$

- $1+1dim$ relativistic models:
  $p_j \rightarrow m \sinh \theta_j \;, \quad S \rightarrow S(\theta_j - \theta_k) \;.$
- $\mathcal{N}=4$ SYM:
  $p_j \rightarrow i \ln \left(\frac{x_d(u_j - i/g)}{x_d(u_j + i/g)} \right) \;, \qquad S \rightarrow \left(\frac{u_j - u_k + i2/g}{u_j - u_k - i2/g} \right) (\sigma(u_j, u_k))^2 \;.$

Possible solutions to the wrapping problem:

- Lüscher corrections (Bajnok-Janik),
- The thermodynamic Bethe Ansatz method.

Al.B. Zamolodchikov: there are two distinct ways to develop a Hamiltonian approach to an Euclidean field theory on a torus geometry.



Periodicity in the R-direction leads to the quantisation of momenta. The states evolve in the time-like L-direction under the influence of Hamiltonian $H_R = H_m$ (m $\equiv$ mirror) .
The partition function is

$$Z(R, L) = \text{Trace}[e^{-LH_R}].$$

Therefore,

$$\frac{1}{L} \quad \sim \quad \text{Temperature.}$$

In the limit $R \to \infty$ the (density of) free energy is

$$Lf^m(L) = \lim_{R\to\infty} -\frac{1}{R} \ln Z(R, L).$$

Quantisation in the L-direction means states evolving in the R direction under the influence of $H_L = H_d$ (d $\equiv$ direct). At large R , in this picture, we get

$$Z(R, L) \sim e^{-RE_0^d(L)}.$$

Therefore, in this limit

$$E_0^d(L) = Lf^m(L).$$

Starting from the Bethe-Yang equations, the TBA method leads to $f^m(L)$:

$$Lf^m(L) = E_0^d(L) = -\frac{m}{2\pi}\int d\theta \cosh\theta \ln(1+e^{-\varepsilon(\theta)}),$$

with

$$\varepsilon(\theta) = mL\cosh\theta - \frac{1}{2\pi}\int d\theta' \phi(\theta-\theta')\ln(1+e^{-\varepsilon(\theta')}),$$

($\varepsilon =$ pseudoenergy) and

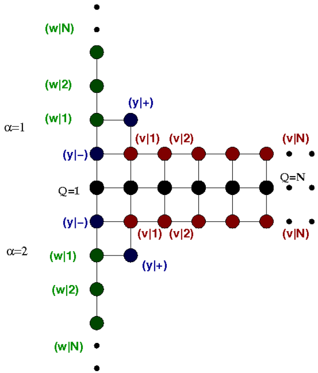
$$\phi(\theta) = -i\frac{d}{d\theta}\ln S(\theta), \quad Y(\theta) = e^{\varepsilon(\theta)}.$$

More than one particle species:

$$m\cosh\theta \to m_A\cosh\theta \ , \quad \phi(\theta) \to \phi_{AB}(\theta) \ , \ \varepsilon(\theta) \to \varepsilon_A(\theta) \ .$$

Some of the m_A 's can be $= 0$: ‘magnonic nodes’.
Excited state TBA [1996]: BLZ, DT.

- Arutyunov-Frolov: Mirror Beisert-Staudacher ABA + string hypotesis;
- Bombardelli-Fioravanti-RT, Gromov-Kazakov-Kozak-Vieira, Arutyunov-Frolov: TBA.



$$\{\varepsilon_N, \varepsilon^{(\alpha)}_{(v|N)}, \varepsilon^{(\alpha)}_{(v|N)}, \varepsilon^{(\alpha)}_{(y,-)}, \varepsilon^{(\alpha)}_{(y,+)}\}$$

Non-standard properties of the $\mathcal{N}=4$ TBA $\leftrightarrow$ presence, in the S-matrix elements, of

$$x(u) \equiv x_m(u) = \frac{u}{2} - i\sqrt{1-\frac{u^2}{4}}.$$

In the first Riemann sheet: $Im(x) < 0$ and $x(u) = 1/x^*(u^*)$.

$$x_d(u) = \begin{cases} x(u) & \text{for} \quad Im(u) < 0; \\ 1/x(u) & \text{for} \quad Im(u) > 0. \end{cases}$$

$$E_0(L) = -\sum_{\mathcal{Q}} \int_{\mathbb{R}} \frac{du}{2\pi} \frac{d\tilde{p}^{\mathcal{Q}}}{du} \ln\left(1 + e^{-\varepsilon_{\mathcal{Q}}(u)}\right),$$

with

$$\tilde{p}^{\mathcal{Q}}(u) = g\, x(u - i\mathcal{Q}/g) - g\, x(u + i\mathcal{Q}/g) + i\mathcal{Q}.$$

Q-particles:

$$\varepsilon_{\mathcal{Q}} = L\, E_{\mathcal{Q}} - \mathcal{L}_{\mathcal{Q}'} * \phi^{\Sigma}_{\mathcal{Q}'\mathcal{Q}} + \mathcal{L}^{(\alpha)}_{(v|M)} * \phi_{(v|M),\mathcal{Q}} + \oint_{\bar{\gamma}_o} dz \mathcal{L}^{(\alpha)}_{y}(z)\, \phi_{y,\mathcal{Q}}(z,u).$$

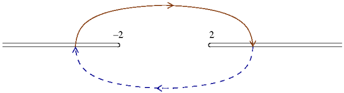
(sum over $\alpha, \mathcal{Q}', M$)

$$E_{\mathcal{Q}}(u) = \ln \frac{x(u - i\mathcal{Q}/g)}{x(u + i\mathcal{Q}/g)}, \;\; \mathcal{L}_A = \ln\left(1 + e^{-\varepsilon_A}\right), \quad Y_A = e^{\varepsilon_A}.$$

The inverse of the temperature $L \in \mathbb{N}$ is quantized for physical reasons, and

$$\mathcal{L} * \phi(u) = \int_{\mathbb{R}} dz\, \mathcal{L}(z)\, \phi(z,u).$$

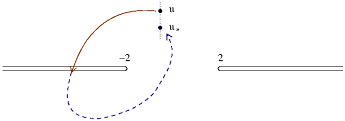
$Y^{(\alpha)}_{y}(u)$ has a pair of square-root branch points at $u = \pm 2$. The integration contour $\bar{\gamma}_o$:



Then

$$Y^{(\alpha)}_{(y|-)}(u) := Y^{(\alpha)}_{y}(u) \quad \text{(first sheet evaluation)}$$

$$Y^{(\alpha)}_{(y|+)}(u) := Y^{(\alpha)}_{y}(u_*) \quad \text{(second sheet evaluation)}$$



$$x_+(u) := x(u_*) = 1/x(u)\ , \quad Im(x_+) > 0.$$

y-particles:

$$\varepsilon^{(\alpha)}_{y}(u) = (\pm i\mu) - \mathcal{L}_{\mathcal{Q}} * \phi_{\mathcal{Q},y}(u) + (\mathcal{L}^{(\alpha)}_{(v|N)} - \mathcal{L}^{(\alpha)}_{(w|M)}) * \phi_M(u).$$

v-particles:

$$\varepsilon^{(\alpha)}_{(v|K)}(u) = -\mathcal{L}_{\mathcal{Q}} * \phi_{\mathcal{Q},(v|K)}(u) + \mathcal{L}^{(\alpha)}_{(v|M)} * \phi_{MK}(u) + \oint_{\bar{\gamma}_o} dz\, \mathcal{L}^{(\alpha)}_{y}(z)\, \phi_K(z-u).$$

w-particles:

$$\varepsilon^{(\alpha)}_{(w|K)}(u) = \mathcal{L}^{(\alpha)}_{(w|M)} * \phi_{MK}(u) + \oint_{\bar{\gamma}_o} dz\, \mathcal{L}^{(\alpha)}_{y}(z)\, \phi_K(z-u).$$

ϕ_M and ϕ_{MN} are relativistic-like kernels:

$$\phi_M(u) = \frac{M/g}{\pi((u)^2 + (M/g)^2)},$$

$$\phi_{MN} = \phi_{|M-N|} + 2\phi_{|M-N|+2} + \cdots + 2\phi_{M+N-2} + \phi_{M+N}.$$

The remaining kernels depends on the two arguments separately:

$$S_{y,\mathcal{Q}}(u,z) = S_{\mathcal{Q},y}(z,u) = \left(\frac{x(z-\frac{i}{g}\mathcal{Q}) - x(u)}{x(z+\frac{i}{g}\mathcal{Q}) - x(u)}\right)\sqrt{\frac{x(z+\frac{i}{g}\mathcal{Q})}{x(z-\frac{i}{g}\mathcal{Q})}},$$

$$S^{\Sigma}_{\mathcal{Q}\mathcal{Q}'}(u,z) = (S_{\mathcal{Q}\mathcal{Q}'}(u-z))^{-1}(\Sigma^{\mathcal{Q}\mathcal{Q}'}(u,z))^{-2},$$

where $\Sigma^{\mathcal{Q}\mathcal{Q}'}$ is the dressing factor for the mirror theory.

- The Y-system fails to be a local functional relation for $Y_{(y|+)}(u)$:

$$\ln \frac{Y^{(\alpha)}_{(y|-)}(u)}{Y^{(\alpha)}_{(y|+)}(u)} = -\sum_{\mathcal{Q}} \int_{\mathbb{R}} dz\, \mathcal{L}_{\mathcal{Q}}(z)(\phi_{\mathcal{Q},(y|-)}(z,u) - \phi_{\mathcal{Q},(y|+)}(z,u)),$$

with

$$\phi_{\mathcal{Q},(y|-)}(z,u) = \phi_{\mathcal{Q},y}(z,u)\ ,\ \ \phi_{\mathcal{Q},(y|+)}(z,u) = \phi_{\mathcal{Q},y}(z,u_*)\ .$$

- The functions $Y_A(u) = e^{\varepsilon_A(u)}$ live on multi-sheeted coverings of the complex plane with an ∞ number of square-root branch points.

Function	Singularity position	
$Y^{(\alpha)}_{y}(u)$	$u = \pm 2 + i\frac{2J}{g}$,	$J = 0, \pm 1, \pm 2, \ldots$
$Y^{(\alpha)}_{(w\|M)}(u)$	$u = \pm 2 + i\frac{J}{g}$,	$J = \pm M, \pm(M+2), \pm(M+4), \ldots$
$Y^{(\alpha)}_{(v\|M)}(u)$		
$Y_M(u)$		

- For the mirror $\mathcal{N} = 4$ SYM: all the cuts are parallel to the real axis and external to the strip $|\Re e(u)| < 2$.
- TBA $\rightarrow$ Y-system: information lost! (i.e. dressing factor, sqrt-related info...)
- Y-system $\rightarrow$ TBA: information on the singularities in $|Im(u)| \leq 1/g$ should be independently supplied (Y-system will propagate it through the whole u -plane)
- Problem: the discontinuity functions depend non locally on the pseudoenergies!!

Define

$$\Delta(u) = \ln \frac{Y_1(u + \frac{i}{g})}{Y_1(u_* + \frac{i}{g})}.$$

This function encodes the branching properties of $Y_1(u)$ around $u = -2 + i/g$:

$$\Delta(u) = \sum_{N,\alpha} \int_{\mathbb{R}} dz \, L^{(\alpha)}_{(v|N)}(z)(K(z + \frac{i}{g}N, u) + K(z - \frac{i}{g}N, u))$$

$$+ \sum_{\alpha} L^{(\alpha)}_{y} *_{\bar{\gamma}_\circ} K(u) + \ldots$$

TBA $\equiv$ dispersion relation

Cavaglià-Fioravanti-RT [arXiv:1005.3016]

Set

$$T(z,u) = \frac{1}{\sqrt{4-z^2}(z-u)},$$

then

$$\frac{\ln\left(Y^{(\alpha)}_{(y|-)}/Y^{(\alpha)}_{(y|+)}\right)}{\sqrt{4-u^2}} = -\sum_{\mathcal{Q}} \int_{\mathbb{R}} \frac{dz}{2\pi i}\, \mathcal{L}_{\mathcal{Q}}(z)(T(z-i\mathcal{Q}/g,u) - T(z+i\mathcal{Q}/g,u)).$$

Denoting by $[f]_Z$ the discontinuity of $f(z)$ on the semi-infinite segments $z = u + iZ/g$ with $u \in (-\infty,-2) \cup (2,\infty)$:

$$[f]_Z = \lim_{\epsilon \to 0^+} f(u+iZ/g+i\epsilon) - f(u+iZ/g-i\epsilon).$$

Result: equivalence with the discontinuity relations

$$\left[\ln\left(Y^{(\alpha)}_{(y|-)}/Y^{(\alpha)}_{(y|+)}\right)\right]_{\pm 2N} = -\sum_{\mathcal{Q}=1}^{N}\left[\ln\left(1+\frac{1}{Y_{\mathcal{Q}}}\right)\right]_{\pm(2N-\mathcal{Q})}.$$

Notice that

$$\frac{\ln\left(Y^{(\alpha)}_{(y|-)}(u)/Y^{(\alpha)}_{(y|+)}(u)\right)}{\sqrt{4-u^2}},$$

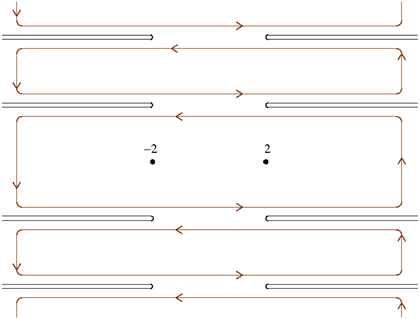
is analytic at the points $u=\pm 2$, but it still has an infinite set branch points at $u=\pm 2\pm i2N/g$ with $N\in\mathbb{N}^*$.
Working hypothesis for the ground-state:

- These are the only singularities on the first Riemann sheet.
- $\ln Y^{(\alpha)}_{(y|-)}/Y^{(\alpha)}_{(y|+)} \to O(1)$ uniformly as $|u|\to\infty$
- The functions $\mathcal{L}_Q = \ln(1+\frac{1}{Y_Q})$ are free of singularities and tend to zero for $\Re e(u)\to\pm\infty$ in the strip $|Im(u)|<Q/g$.

Cauchy's theorem:

$$\frac{\ln\left(Y^{(\alpha)}_{(y|-)}(u)/Y^{(\alpha)}_{(y|+)}(u)\right)}{\sqrt{4-u^2}} = \oint_\gamma \frac{dz}{2\pi i}\frac{\ln\left(Y^{(\alpha)}_{(y|-)}(z)/Y^{(\alpha)}_{(y|+)}(z)\right)}{(z-u)\sqrt{4-z^2}}.$$

γ : positive-oriented contour inside the strip $|Im(u)| < 1/g$.
Then, deform γ into the homotopically equivalent contour Γ_O , union of an infinite number of rectangular contours:



Since $\ln Y^{(\alpha)}_{(y|-)}/Y^{(\alpha)}_{(y|+)} \to O(1)$ uniformly as $|u| \to \infty$ the sum of vertical segment contributions vanish as the horizontal size of the rectangular contours tends to ∞ and using the discontinuity relation:

$$-\sum_{K=1}^{\infty}\sum_{s}\left(\int_{\mathbb{R}-i2sK/g+i\epsilon}-\int_{\mathbb{R}-i2sK/g-i\epsilon}\right)\sum_{\mathcal{Q}=1}^{K}\mathcal{L}_{\mathcal{Q}}(z+is\mathcal{Q}/g)\frac{dz}{2\pi i\sqrt{4-z^2}(z-u)},$$

with $s=\pm 1$. Further, several cancelations take place and with a change of variables we get

$$\sum_{Q,s}^{\infty}\int_{\mathbb{R}+is(Q/g-\epsilon)}\mathcal{L}_Q(z)\frac{dz}{2\pi i\sqrt{4-(z+isQ/g)^2}(z+isQ/g-u)}.$$

Using the hypothesis that for the ground-state the functions $\mathcal{L}_Q$ are free of singularities and tend to zero for $\Re e(u) \to \pm\infty$ in the strip $|Im(u)| < Q/g$.

$$-\sum_Q \int_{\mathbb{R}} \mathcal{L}_Q(z) \left(\frac{1}{\sqrt{4-(z-iQ/g)^2}(z-iQ/g-u)} - \frac{1}{\sqrt{4-(z+iQ/g)^2}(z+iQ/g-u)} \right) \frac{dz}{2\pi i},$$

which matches perfectly the TBA result!

The derivation of the remaining TBA equations from the functional equations for the discontinuities is very similar...

Extended Y-system

Y-system+

$$[\Delta]_{\pm 2N} = \mp \sum_{\alpha=1,2} \left(\left[\ln\left(1+\frac{1}{Y^{(\alpha)}_{(y|\mp)}}\right)\right]_{\pm 2N} + \sum_{M=1}^{N} \left[\ln\left(1+\frac{1}{Y^{(\alpha)}_{(v|M)}}\right)\right]_{\pm(2N-M)} + \ln\left(\frac{Y^{(\alpha)}_{(y|-)}}{Y^{(\alpha)}_{(y|+)}}\right)\right),$$

$$\left[\ln\left(\frac{Y^{(\alpha)}_{(y|-)}}{Y^{(\alpha)}_{(y|+)}}\right)\right]_{\pm 2N} = -\sum_{Q=1}^{N} \left[\ln\left(1+\frac{1}{Y_Q}\right)\right]_{\pm(2N-Q)},$$

$$\left[\ln Y^{(\alpha)}_{(w|1)}\right]_{\pm 1} = \ln\left(\frac{1+1/Y^{(\alpha)}_{(y|-)}}{1+1/Y^{(\alpha)}_{(y|+)}}\right), \qquad \left[\ln Y^{(\alpha)}_{(v|1)}\right]_{\pm 1} = \ln\left(\frac{1+Y^{(\alpha)}_{(y|-)}}{1+Y^{(\alpha)}_{(y|+)}}\right).$$

Conclusions

- The local form of the Y-system is not convertible to TBA without knowledge on the discontinuity functions in the strip $|Im(u)| \leq 1/g$.
- The discontinuity information is encoded in a local set of functional relations.
- The ground-state TBA can be derived, including the dressing factor, from the extended Y-system assuming no extra singularities are present on the reference sheet.
- Conjecture: all the proposed excited state TBA equations can be obtained in the same way by including extra *log* singularities.