Integrability and Non-planarity

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arXiv:1005:2611 [hep-th] (P. Caputa, C.K. and K. Zoubos)
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Outline

- Integrability of the spectral problem of planar $\mathcal{N} = 4$ SYM
- Beyond the planar limit for $\mathcal{N} = 4$ SYM
- Non-planar ABJ(M), integrability and parity
- $\mathcal{N} = 4$ SYM with gauge group $SO(N)$
- Summary and outlook

The spectral problem of planar $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM, gauge group SU(N) $\longleftrightarrow$ IIB strings on $AdS_5 \times S^5$

$$\underbrace{\lambda = g_{\mathrm{YM}}^2 N,}_{\text{loop expansion}} \qquad \underbrace{\frac{1}{N}}_{\text{topological exp.}} \qquad \underbrace{\frac{R^2}{\alpha'} = \sqrt{\lambda},}_{\text{spectrum}} \quad \underbrace{g_s = \frac{\lambda}{N}}_{\text{interactions}}$$

Local gauge invariant operators $\longleftrightarrow$ string states
Conformal dimensions, $\Delta \longleftrightarrow$ energies of string states

The spectral problem of $\mathcal{N} = 4$ SYM:
Determine $\Delta = \Delta(\lambda, N) \Leftrightarrow$ Diagonalize dilatation generator D

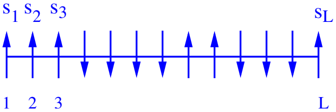
The planar version: $N \to \infty$, integrable

Theme of the talk: What happens when we go beyond the planar limit (i.e. N finite)

Integrability of the planar spectral problem

Ex: SU(2) sector, one loop order, $\mathcal{O} = \mathrm{Tr}(ZZZXXXXZZXXXZ)$

[Minahan &Zarembo '02]



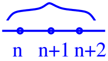
$S_{L+m} = S_m$

$$\hat{D} = \frac{\lambda}{2}\sum_{n=1}^{L}(1 - \bar{\sigma}_n \cdot \bar{\sigma}_{n+1}) = \lambda \sum_{n=1}^{L}\left(1 - P_{n,n+1}\right) \equiv \lambda \sum_{n=1}^{L} \hat{H}_{n,n+1}$$

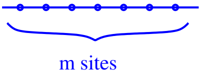
Conserved charges: $\exists \ \hat{Q}_i, \qquad i = 1, \ldots, L: \qquad \left[\hat{Q}_i, \hat{Q}_j\right] = 0$

$$\hat{Q}_1 = \sum_n e^{i\hat{P}_n}, \qquad \hat{Q}_2 = \hat{D}$$

$$\hat{Q}_3 = \sum_n [\hat{H}_{n,n+1}, \hat{H}_{n+1,n+2}] =$$



$\hat{Q}_m$:



Bethe equations

Length L with M excitations: M Bethe equations for $\{u_k\}_{k=1}^M$

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j=1, j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \qquad k = 1, \ldots, M$$

Eigenvalues for $\hat{D}$

$$E(\{u_k\}) = \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}}$$

Cyclicity constraint

$$\sum_{k=1}^{M} p_k = 0, \qquad \text{where} \qquad u_k = \frac{1}{2} cot \left(\frac{p_k}{2}\right)$$

Beyond one-loop order

Higher orders in λ:

Spin chain with long range interactions

Order λ^n: interactions between $n+1$ nearest neighbours

Still integrable:

$$\exists \text{ conserved charges } Q_i,\ i = 1, \ldots, L:$$

$$\text{at } n\text{-loop order: } Q_i = Q_i^0 + \lambda Q_i^1 + \ldots + \lambda^n Q_i^n,$$

$$\left[Q_i, Q_j\right] = \mathcal{O}(\lambda^{n+1}), \qquad Q_i^n \text{ of range } (i+n)$$

Conjectured to be true at any loop order (proved to 2-4 loops)

[Beisert, C.K. Staudacher '03, Beisert, & Staudacher '05, Beisert, Eden Staudahcer '06, Beisert, Hernandez, Lopez '06, ...]

Discovery: Observation of otherwise unexplained degeneracies in the spectrum [Beisert, C.K. & Staudacher '03]

Parity I

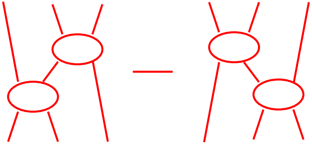
$$\hat{P}\mathrm{Tr}(Z^3X^2ZX) = \mathrm{Tr}(XZX^2Z^3) = \mathrm{Tr}(Z^3XZX^2), \qquad \hat{P}^2 = 1$$

$[\hat{P}, \hat{H}] = 0$, i.e. eigenstates of $\hat{H}$ can be chosen of definite parity, $P = \pm 1$

Observation: Pairs of operators with opposite parity but the same energy. Survive loop corrections.

Explanation: The existence of $\hat{Q}_3$, i.e. integrability

$$Q_3 = \sum_n [H_{n,n+1}, H_{n+1,n+2}] =$$



$\{\hat{Q}_3, P\} = 0, \qquad [\hat{Q}_3, \hat{H}] = 0 \implies$

The operators in a degenerate pair are connected via $\hat{Q}_3$.

Parity II

Bethe eqns, dispersion relation, cyclicity constraint invariant under $\{u_k\} \to \{-u_k\}$.

Unpaired solutions: $|\{u_k\}\rangle$ such that $\{u_k\} = \{-u_k\}$

Paired solutions: $|\{u_k\}\rangle, |\{-u_k\}\rangle$ where $\{u_k\} \neq \{-u_k\}$.

Parity in general:

$$\hat{P}|\{u_k\}\rangle = (-1)^{M(L+1)}|\{-u_k\}\rangle$$

Unpaired solutions: $P = (-1)^{M(L+1)}$

Paired solutions can be combined to parity eigenstates:

$$\hat{P}(|\{u_k\}\rangle \pm |\{-u_k\}\rangle) = (-1)^{M(L+1)}(\pm 1)(|\{u_k\}\rangle \pm |\{-u_k\}\rangle)$$

[A. Ipsen]

Beyond the planar limit for $SU(N)$ $\mathcal{N}=4$ SYM

$\mathcal{O} = \mathrm{Tr}(X\ldots XZ\ldots)\mathrm{Tr}(X\ldots XZ\ldots) \subset SU(2)$ sector.

[Constable et al '02], [Beisert, C.K., Plefka, Semenoff & Staudacher '02]

$$\hat{D} = -g_{YM}^2 : \mathrm{Tr}[Z,X][\check{Z},\check{X}] :, \qquad (\check{Z})_{\alpha\beta} = \frac{\delta}{\delta Z_{\beta\alpha}}$$

$$= \lambda\Big(D_0 + \underbrace{\frac{1}{N}D_+}_{\text{adds a trace}} + \underbrace{\frac{1}{N}D_-}_{\text{removes a trace}}\Big)$$

Origin: Quartic interaction between scalars

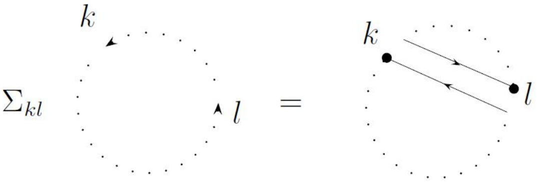
Example:

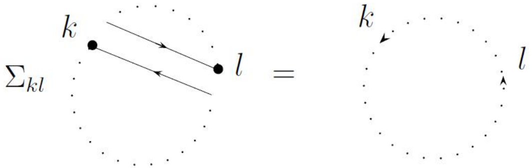
$$\mathrm{Tr}(ZX\check{Z}\check{X})\cdot\mathrm{Tr}(XZXXZ)\,\mathrm{Tr}(XZ) = \mathrm{Tr}(ZX\check{Z}ZXXZ)\,\mathrm{Tr}(XZ)$$

$$= N\mathrm{Tr}(ZXXXZ)\,\mathrm{Tr}(XZ) + \mathrm{Tr}(ZX)\,\mathrm{Tr}(ZXX)\,\mathrm{Tr}(XZ) + \mathrm{Tr}(ZXZZZXXZ)$$

The non-planar part of $\hat{D}$

$$D_+ + D_- = \sum_k \sum_{l \neq k+1} \left(1 - P_{k,l}\right) \Sigma_{k+1,l} \equiv \sum_k H_k^{(1)}$$





Search for integrability beyond the planar limit

- Search for conserved charges (extremely non-local, involve trace splitting and joinning)
- Search for S-matrix and/or Bethe ansatz (Hilbert space enormously complicated)

$\frac{1}{N}$-corrections to short operators

Easy to evaluate

- $D_+\mathcal{O}$, $D_-\mathcal{O}$ involves a finite (small) number of operations
- Only diagonalization of finite-dim. matrix

Strategy:

- Consider closed set of operators. Ex: Length 8 with 3 excitations
- Find the planar eigenvalues and eigenstates (can be checked by Bethe eqns.).
- Write down $\hat{D}$ in the basis of planar eigenstates and do perturbation theory in $\frac{1}{N}$.

Lessons learned

- Including $\frac{1}{N}$ corrections, degeneracies between parity pairs are lifted, but still $[H, P] = 0$
  $\implies$ absence of Q_3, at least in its previous form
  [Beisert, C.K. Staudacher '03]
- Δ does not always have a well-defined expansion in λ and $\frac{1}{N}$ but D has. (Higher loop effect.)
  [Ryzhov '01], [Arutyunov et al. '02] [Bianchi, Kovacs Rossi,Stanev '02] [Beisert, C.K. Staudacher '03]
- Degeneracies between single and double trace states (of equal parity) lead to $\frac{1}{N}$ as opposed to $\frac{1}{N^2}$ corrections.

Lessons learned for ABJM and ABJ theory

- ABJM: Including $\frac{1}{N}$-corrections, degeneracies between parity pairs are lifted, but still $[H, P] = 0$
  $\Longrightarrow$ absence of Q_3, at least in its previous form [C.K., Orselli Zoubos '08]
- ABJ: Including non-planar corrections, $[H, P] \neq 0$ (and degeneracies are lifted). [Caputa, C.K., Zoubos '09]

Search for integrability beyond the planar limit

- Search for conserved charges (extremely non-local, involve trace splitting and joinning)
- Search for S-matrix and/or Bethe ansatz (Hilbert space enormously complicated)

$\mathcal{N} = 4$ SYM with gauge group $SO(N)$

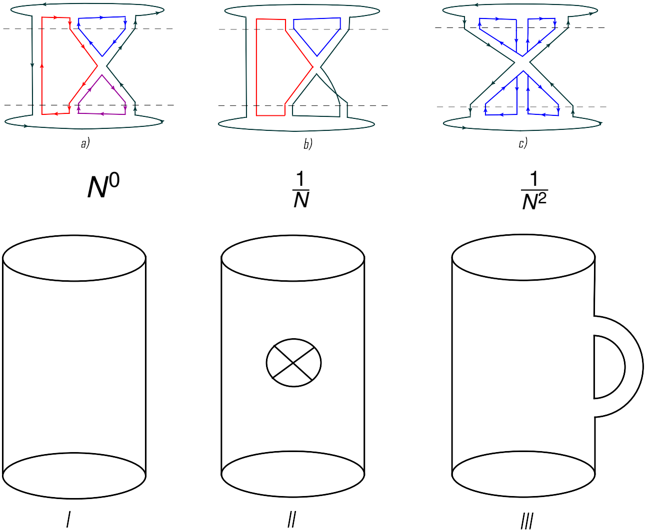
$\mathcal{N} = 4$ SYM, gauge group SO(N) $\longleftrightarrow$ IIB strings on $AdS_5 \times RP^5$
[Witten '98]

$RP^5 = S^5/Z_2, \quad (\sum_{i=1}^{6} X_i^2 = 1, X^i \equiv -X^i)$, orientifold

Feynman diags w/ cross-caps $\longleftrightarrow$ non-orientable world sheets

Leading $\frac{1}{N}$-effects do not involve chain splitting and joining

Weighting of Feynman and string diagrams



The planar spectral problem for $SO(N)$

Restrict to $SU(2)$ sector: $\mathcal{O} = \mathrm{Tr}(X \ldots XZ \ldots)$

Parity is gauged: $X^T = -X \implies$

$$\hat{P}\mathrm{Tr}(X_{i_1} \ldots X_{i_L}) = \mathrm{Tr}(X_{i_L} X_{i_{L-1}} \ldots X_{i_1}) = (-1)^L \mathrm{Tr}(X_{i_1} \ldots X_{i_L})$$

Planar dilatation operator at one-loop order

$$\hat{D}_0^{SO(N)} = \frac{\lambda}{2} \sum_{i=1}^{L} (1 - P_{i,i+1}) = \frac{1}{2} \hat{D}_0^{SU(N)}$$

Planar spectral problem $\subset$ planar spectral problem for $SU(N)$

Surviving states:

- One state from each parity pair: $|\{u_k\}\rangle + (-1)^{M(L+1)+L} |\{-u_k\}\rangle$
- Unpaired states of L and M even

$\frac{1}{N}$ effects for gauge group $SO(N)$

Restrict to $SU(2)$ sector: $\mathcal{O} = \mathrm{Tr}(X \ldots XZ \ldots)\mathrm{Tr}(X \ldots XZ \ldots)$

$$\hat{D} = -\frac{g_{YM}^2}{8\pi^2}\mathrm{Tr}[Z,X][\check{Z},\check{X}], \qquad (\check{Z})_{\alpha\,\beta} Z_{\gamma\epsilon} = \frac{1}{2}(\delta_{\alpha\epsilon}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\epsilon})$$

$$= \frac{\lambda}{2}(D_0 + \frac{1}{N}D_+ + \frac{1}{N}\tilde{D}_- + \underbrace{\frac{1}{N}D_{flip}}_{\text{Acts inside a single trace}})$$

$$D_{flip} \cdot \mathrm{Tr}(XWZY) = \mathrm{Tr}([Z,X]W^TY) + \mathrm{Tr}([Z,X]YW^T)$$

OBS: D_{flip} involves a sum of such contractions

Leading energy corrections of order $\frac{1}{N}$: $\quad E_1 = \langle \mathcal{O}|D_{flip}|\mathcal{O}\rangle$

We consider only the perturbation D_{flip}

Why interesting:

- leading $1/N$ corrections at strong coupling only due to D_{flip}
- valid for single trace states, not degenerate with multi-trace states
- defines a new type of spin chain interaction

How

- Try to construct conserved charges $Q = Q^0 + \frac{1}{N}Q^1$

$$0 = [D_0, Q^1] + [D_{flip}, Q^0], \qquad \text{did not succeed}$$

- Try to look for perturbed Bethe equations

Searching for a perturbed Bethe ansatz for D_{flip}

Strategy

- Determine analytically the leading $\frac{1}{N}$ correction for two-excitation states by QM perturbation theory.
- Construct/guess a perturbed Bethe ansatz which reproduces these corrections.
- Determine numerically the leading contribution to states with more excitations by explicit diagonalization and check whether the perturbed Bethe ansatz gives the correct result.

Leading $\frac{1}{N}$ corrections for $SO(N)$, considering only D_{flip}

Two excitation states: $O_p^J = \mathrm{Tr}(XZ^pXZ^{J-p})$, J even, $L = J + 2$

Planar eigenstates: $D_0|n^J\rangle = E_n^0|n^J\rangle$

$$|n^J\rangle = \frac{1}{J+1}\sum_{p=0}^{J}\cos\left(\frac{\pi n(2p+1)}{J+1}\right)O_p^J, \qquad 0 \le n \le \frac{J}{2}$$

$$E_n^0 = 2\sin^2\left(\frac{\pi n}{J+1}\right)$$

Non-planar correction: $E_n = E_n^0 + \frac{1}{N}E_n^{flip} = E_n^0 + \frac{1}{N}\langle n^J|D_{flip}|n^J\rangle$

$$\begin{aligned} E_n^{flip} &= -\sin^2(\frac{\pi n}{J+1}) \\ &\quad -\frac{1}{J+1}\left\{2\tan^2(\frac{\pi n}{J+1}) - \frac{1}{2}\tan^2(\frac{2\pi n}{J+1})\cos(\frac{2\pi n}{J+1})\right\} \end{aligned}$$

OBS: Analytical result, prediction for string theory

[Beisert, Dippel.,Staudacher '04]

Correction of Bethe equations (i.e. correction of momenta)

$$\left(\frac{x(u_k+\frac{i}{2})}{x(u_k-\frac{i}{2})}\right)^L = \prod_{j\neq k}^M \left(\frac{u_k-u_j+i}{u_k-u_j-i}\right)$$

$$x(u) = u(1 - g^2\frac{1}{u^2}), \qquad e^{ip} = \frac{x(u+\frac{i}{2})}{x(u-\frac{i}{2})}, \qquad g^2 = \frac{g_{YM}^2 N}{8\pi^2}$$

Correction of dispersion relation

$$E(\{p_k\}) = \sum_k 4\sin^2\left(\frac{p_k}{2}\right) - 16g^2\sin^4\left(\frac{p_k}{2}\right)$$

Two-excitation states with $M=2$, $L=J+2$: $E = E_0 + g^2\delta E$

$$\delta E = -\underbrace{16\sin^4\left(\frac{n\pi}{J+1}\right)}_{\text{corr. of disp. rel.}} - \underbrace{64\frac{1}{J+1}\cos^2\left(\frac{n\pi}{J+1}\right)\sin^4\left(\frac{n\pi}{J+1}\right)}_{\text{correction of momenta}}$$

Leading $\frac{1}{N}$ corrections for $SO(N)$, considering only D_{flip}

$$E_n^{flip} = \underbrace{-\sin^2\left(\frac{\pi n}{J+1}\right)}_{\text{corr. of disp. rel.?}} \underbrace{-\frac{1}{J+1}\left\{2\tan^2\left(\frac{\pi n}{J+1}\right) - \frac{1}{2}\tan^2\left(\frac{2\pi n}{J+1}\right)\cos\left(\frac{2\pi n}{J+1}\right)\right\}}_{\text{correction of momenta?}}$$

E_n^{flip} from perturbed Bethe equations?

Correction of dispersion relation

$$E(\{p_k\}) = \sum_k 2\sin^2\left(\frac{p_k}{2}\right) - \frac{1}{N}\sin^2\left(\frac{p_k}{2}\right)$$

Correction of Bethe equations (i.e. correction of momenta)

$$\left(\frac{x(u_k+\frac{i}{2})}{x(u_k-\frac{i}{2})}\right)^L = \prod_{j\neq k}^M \left(\frac{u_k-u_j+i}{u_k-u_j-i}\right), \qquad e^{ip} = \frac{x(u+\frac{i}{2})}{x(u-\frac{i}{2})}$$

Parametrizing $x(u) = u(1 - \frac{1}{N}f(u))$ we find

$$f(u+\tfrac{i}{2}) - f(u-\tfrac{i}{2}) = -i\frac{1}{(16u^3)(4u^2-1)}.$$

Use this to make predictions for states with $L = 8$, $M = 4$

Find energy corrections by diagonalization and compare

Conclusion: Does not work.

E_n^{flip} from another perturbed Bethe ansatz

Correction of dispersion relation

$$E(\{p_k\}) = \sum_k 2\sin^2\left(\frac{p_k}{2}\right) - \frac{1}{N}\sin^2\left(\frac{p_k}{2}\right)$$

Correction of Bethe equations by new phase factor

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L = \prod_{j\neq k}^{M}\left(\frac{u_k-u_j+i}{u_k-u_j-i}\right)\left(1+\frac{i}{N}h(u_k-u_j)\right), \qquad e^{ip} = \frac{u+\frac{i}{2}}{u-\frac{i}{2}}$$

From our analytical solution for two-excitation states we find

$$h(u) = \frac{1}{2u^3(u^2-1)}$$

Use this to make predictions for states with $L=8$, $M=4$

Find energy corrections by diagonalization and compare

Conclusion: Does not work.

Summary and outlook

- Have been able to look for integrability beyond the planar limit by conventional methods. No sign of integrability found (yet?)

  Need to rethink the concept of integrability when going beyond the planar limit (or forget it?)

- Some analytical results on non-planar anomalous dimensions for $\mathcal{N} = 4$ SYM with gauge $SO(N)$. Predictions for string theory.

  Dual string theory not yet studied systematically: Spinning strings, pp-wave strings,...