Integrability of high energy scattering amplitudes in $N = 4$ SUSY

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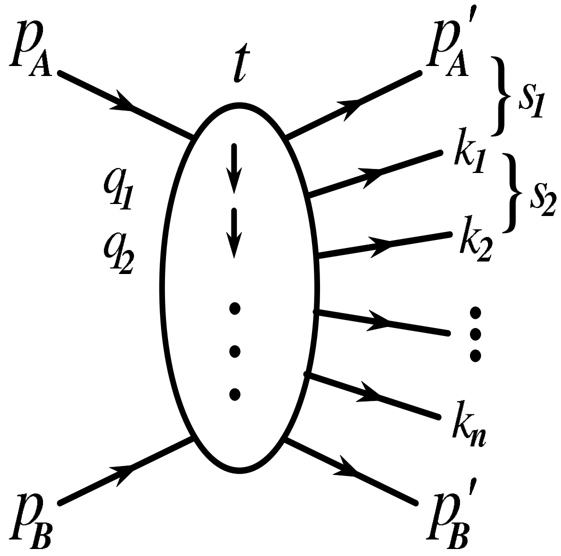
1. Multi-Regge processes
2. Integrability of the BFKL dynamics
3. BDS amplitudes at large energies
4. Steinmann relations and Mandelstam cuts
5. Factorization and exponentiation
6. Integrable open Heisenberg spin chain
7. Composite states of several reggeized gluons
8. Discussion

Reference: **J.Phys.A42:304020,2009, hep-th: 0902.1444**

1 Amplitudes in multi-Regge kinematics



$$M^{BFKL}_{2\to 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2 c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1}c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

Reggeon-Reggeon-gluon vertex

$$C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \omega_r = -\frac{\alpha_s N_c}{2\pi}\left(\ln\frac{|q_r^2|}{\mu^2} - \frac{1}{\epsilon}\right)\,,\ \sigma_t = \sum_n \int d\Gamma_n\, |M_{2\to 2+n}|^2$$

2 Analyticity, unitarity and bootstrap

Steinmann relations for overlapping channels

$$\Delta_{s_r} \Delta_{s_{r+1}} M_{2\to 2+n} = 0$$

Dispersion representation for $M_{2\to3}$ in the Regge ansatz

$$M_{2\to3} = c_1(-s)^{j(t_2)}(-s_1)^{j(t_1)-j(t_2)} + c_2(-s)^{j(t_1)}(-s_2)^{j(t_2)-j(t_1)}$$

Dispersion representation for $M_{2\to4}$ in the Regge ansatz

$$M_{2\to4} = d_1(-s)^{j_3}(-s_{012})^{j_2-j_3}(-s_1)^{j_1-j_2} + d_2(-s)^{j_1}(-s_{123})^{j_2-j_1}(-s_3)^{j_3-j_2}$$

$$+d_3(-s)^{j_3}(-s_{012})^{j_1-j_3}(-s_2)^{j_2-j_1} + d_4(-s)^{j_1}(-s_{123})^{j_3-j_1}(-s_2)^{j_2-j_3}$$

$$+d_5(-s)^{j_2}(-s_1)^{j_1-j_2}(-s_3)^{j_3-j_2}\,, \; j_r = j(t_r)$$

Bootstrap relation in LLA (BFKL (1975-1978))

$$\pi\omega(t_1)\, M_{2\to2+n} = \sum_r \Im_{s_{0r}} M_{2\to2+n} = \sum_t M_{2\to2+t} M_{2+t\to2+n}$$

3 BFKL dynamics for colorless states

Balitsky-Fadin-Kuraev-Lipatov equation (1975)

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\; \sigma_t \sim s^{\Delta}\;,\;\; \Delta = -\frac{\alpha_s N_c}{2\pi}\,E,$$

Holomorphic separability

$$\mathrm{H}_{12} = h_{12} + h_{12}^{*}\;,\;\; h_{12} = \ln(p_1 p_2) + \tfrac{1}{p_1}\ln(\rho_{12})\,p_1 + \tfrac{1}{p_2}\ln(\rho_{12})\,p_2 - 2\gamma$$

Bartels-Kwiecinski-Praszalowicz equation at $N_c \to \infty$

$$E\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = H\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)\;,\;\; H = \frac{1}{2}\sum_{k=1}^{n} H_{k,k+1} = h + h^{*}$$

Holomorphic factorization and Möbius invariance (L. (1986))

$$\Psi(\vec{\rho}_1,\vec{\rho}_2,...,\vec{\rho}_n) = \sum_{r,s} a_{r,s}\,\Psi_r(\rho_1,...,\rho_n)\,\Psi_s(\rho_1^{*},...,\rho_n^{*})\,,$$

$$M_a^2\Psi_r = m(m-1)\Psi_r\,,\;\; M_a^{*2}\Psi_s = \widetilde{m}(\widetilde{m}-1)\Psi_s\,,\;\; m = \frac{1}{2} + i\nu + \frac{n}{2}$$

4 Integrable closed spin chain

Monodromy and transfer matrices (L. (1993))

$$t(u) = L_1 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} , \; L_k = \begin{pmatrix} u + \rho_k \, p_k & p_k \\ -\rho_k^2 \, p_k & u - \rho_k \, p_k \end{pmatrix} ,$$

$$T(u) = A(u) + D(u) \, , \; [T(u), h] = 0$$

Yang-Baxter equation (L. (1993))

$$t_{r_1'}^{s_1}(u) \, t_{r_2'}^{s_2}(v) \, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u) \, t_{r_2}^{s_2'}(v) \, t_{r_1}^{s_1'}(u) \, , \; \hat{l} = u \, \hat{1} + i \, \hat{P}$$

Duality symmetry (L. (1999))

$$p_r \longrightarrow \rho_{r+1,r} \longrightarrow p_{r+1}$$

Heisenberg spin model (L. (1994); F., K.(1995))

$$\vec{S}_k = (\rho_k \partial_k, \partial_k, -\rho_k^2 \partial_k)$$

5 Elastic BDS amplitude at $s/t \to \infty$

Regge asymptotics at $s/t \to \infty$

$$M_{2\to 2}^{BDS} = \Gamma(t) \left(\frac{-s}{\mu^2}\right)^{\omega(t)} \Gamma(t)$$

Reggeized gluon trajectory

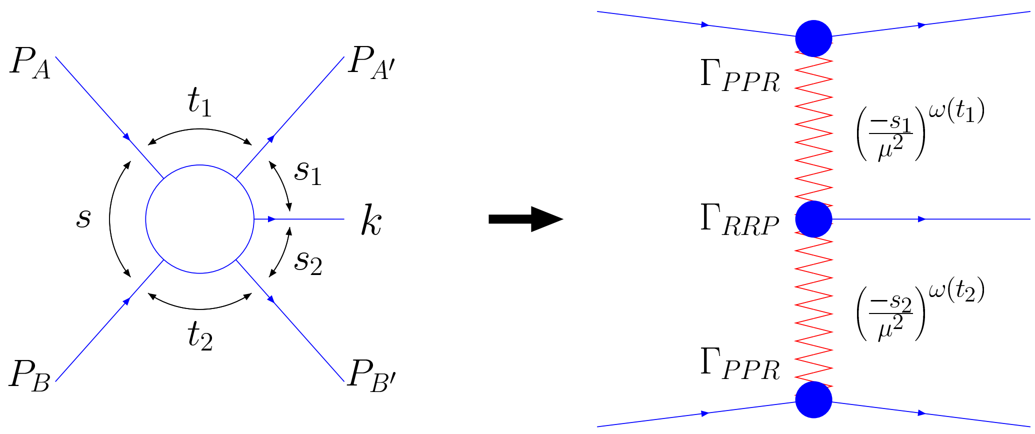
$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln\frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{4\epsilon} + \beta(a')\right) \, , \ \gamma_K(a) = 4a + \ldots$$

Reggeon residues

$$\ln\Gamma(t) = \ln\frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2}\right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2}\zeta_2$$

$$- \int_0^a \frac{da'}{a'} \ln\frac{a}{a'} \left(\frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a')\right)$$

6 One particle production (BLS)

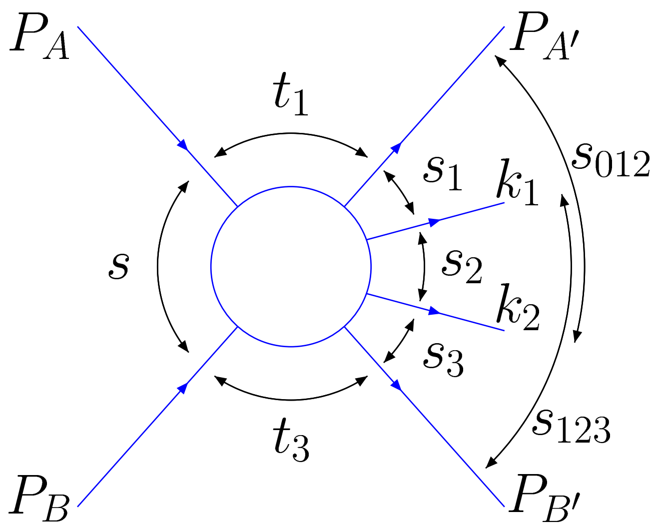


$$M_{2\to3}^{BDS} = \Gamma(t_1)\left(c_1^{12}(-s\kappa_{12})^{\omega_2}(-s_1)^{\omega_{12}} + c_2^{12}(-s\kappa_{12})^{\omega_1}(-s_2)^{\omega_{21}}\right)\Gamma(t_2)\,,$$

$$\kappa_{12} = \frac{s_1 s_2}{s} = |k_a|^2\,,\ c_1^{12} = |\Gamma_{12}|\frac{\sin\pi\omega_{1a}}{\sin\pi\omega_{12}}\,,\ c_2^{12} = |\Gamma_{12}|\frac{\sin\pi\omega_{2a}}{\sin\pi\omega_{21}}\,,$$

$$(\Gamma_{12})_{s,s_1,s_2>0} = |\Gamma_{12}|\exp(i\pi\,\omega_a)\,,\ \omega_a = \frac{\gamma_K}{8}\ \ln\frac{|k_a|^2\lambda^2}{|q_1|^2|q_2|^2}\,,\ \lambda^2 = \mu^2\exp(1/\epsilon)$$

7 Regge factorization violation (BLS)



$$M_{2\to 4}^{BDS}|_{s,s_2>0;\, s_1,s_3<0} = C\,\Gamma_1 \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma_{21} \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)} \Gamma_{32} \left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)} \Gamma_3\,,$$

$$C = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi \left(\ln\frac{t_1 t_3}{(\vec{k}_a+\vec{k}_b)^2\mu^2} - \frac{1}{\epsilon}\right)\right]\,,\ \ \frac{M_{2\to 4}^{pole}}{|M_{2\to 4}^{BDS}|} = e^{-i\pi\omega_2}\cos\pi\omega_{ab}$$

8 Mandelstam cuts in j_2-plane

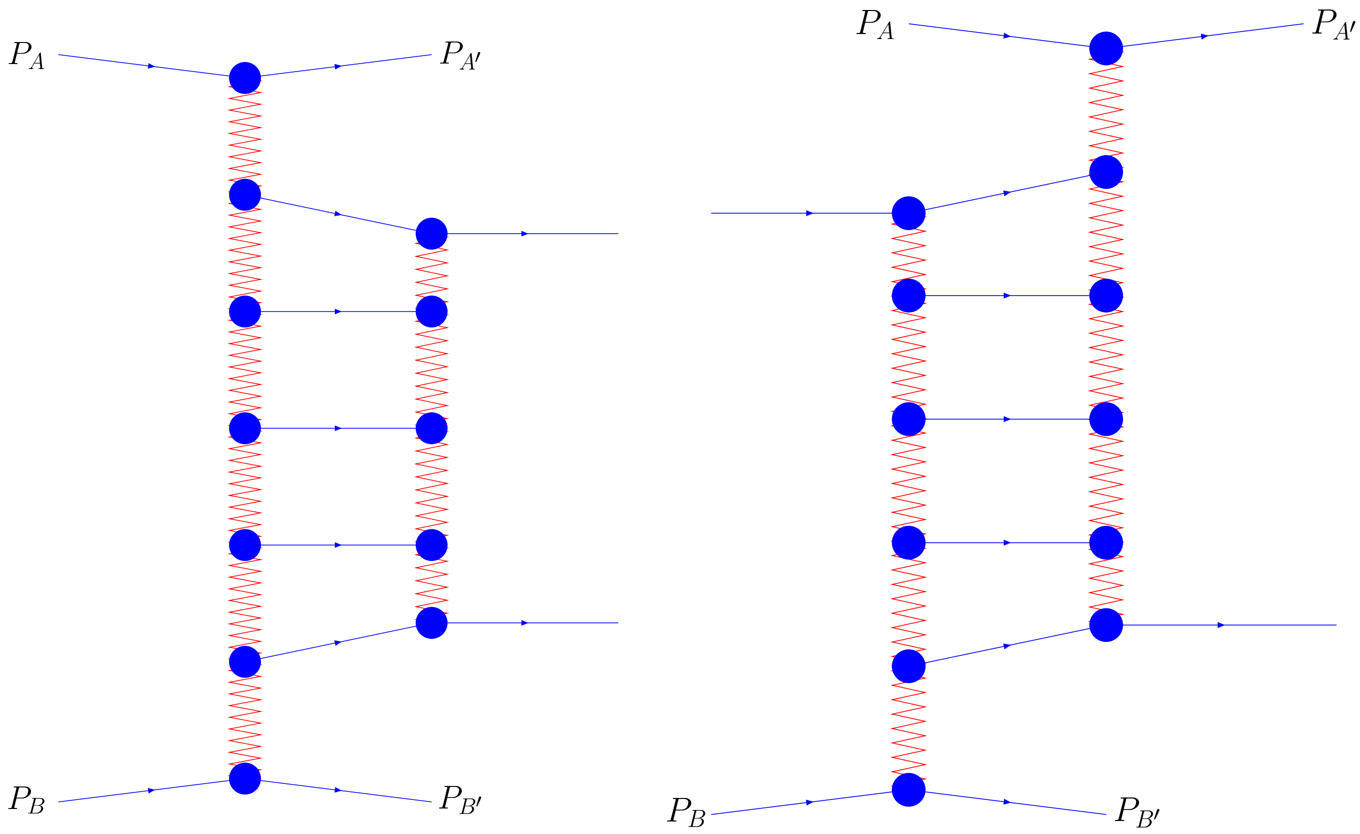


Figure 1: BFKL ladders in $M_{2\to 4}$ and $M_{3\to 3}$

9 Factorization and exponentiation

Analyticity constraint and factorization hypothesis

$$M_{2\to 4} = M_{2\to 4}^{pole} + M_{2\to 4}^{cut} = c\, M_{2\to 4}^{BDS}$$

BDS ansatz at $s, s_2 > 0, s_1, s_3 < 0$

$$M_{2\to 4}^{BDS} = |M_{2\to 4}^{BDS}|\, e^{-i\pi\omega_2} e^{i\delta}\,,\ \ \delta = \frac{\gamma_K}{4}\ \ln\frac{|q_1 q_3 k_a k_b|}{|k_a + k_b|^2 |q_2^2|}$$

Regge pole and cut contributions at $s, s_2 > 0, s_1, s_3 < 0$

$$c\, e^{i\delta} = \cos\pi\omega_{ab} + i\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\,(-s_2)^{\omega}\, f(\omega)\,,\ \ \omega_{ab} = \frac{\gamma_K}{4}\ \ln\frac{|k_a q_3|}{|k_b q_1|}$$

Prediction from the analyticity constraint

$$c - 1 = \frac{\ln s_2}{i\pi}\left(\frac{\delta^2}{2} - \frac{\pi^2\omega_{ab}^2}{2}\right) = -2\pi i\frac{a^2}{4}\ \ln s_2\ \ln\frac{|k_b|^2|q_1|^2}{|k_a + k_b|^2|q_2|^2}\ \ln\frac{|k_a|^2|q_3|^2}{|k_a + k_b|^2|q_2|^2}$$

Factor c is not a phase

10 BFKL equation for octets (BLS)

Regge singularity trajectories

$$\omega(t_2) = -a\left(E + \ln\frac{-t_2}{\mu^2} - \frac{1}{\epsilon}\right)\,,\ \Delta = -aE$$

BFKL hamiltonian for partial waves f_{j_2}

$$H = \ln\frac{|p_1 p_2|^2}{|p_1+p_2|^2} + \frac{1}{2}\,p_1 p_2^* \ln|\rho_{12}|^2 \frac{1}{p_1 p_2^*} + \frac{1}{2}\,p_1^* p_2 \ln|\rho_{12}|^2 \frac{1}{p_1^* p_2} + 2\gamma$$

Eigenfunctions and eigenvalues

$$\Psi_{n,\nu} = \left(\frac{p_1}{p_2}\right)^{i\nu+n/2}\left(\frac{p_1^*}{p_2^*}\right)^{i\nu-n/2}\,,\ E_{n,\nu} = 2Re\,\psi(i\nu+\frac{|n|}{2}) - 2\psi(1)$$

Factorization of infrared divergencies in LLA

$$M^{2\to 4}_{|_{s,s_2>0;s_1,s_3<0}} = (1+i\delta_{2\to 4})\ M^{BDS}_{2\to 4}\,,$$

11 Möbius and conformal invariances

Analytic result in LLA in the region $a \ln s_2 \sim 1$

$$\delta_{2\to 4} = \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \, (V^*)^{i\nu - \frac{n}{2}} \, V^{i\nu + \frac{n}{2}} \left(s_2^{\Delta(\nu,n)} - 1 \right)$$

Duality transformation to the Möbius representation

$$V = \frac{q_3 k_1}{k_2 q_1} \to \frac{z_{03} z_{0'1}}{z_{0'3} z_{01}}$$

Perturbation theory expansion

$$i\delta_{2\to 4} = -2i\pi \, a^2 \ln s_2 \ln \frac{|k_1 + k_2||q_2|}{|k_2||q_1|} \ln \frac{|k_1 + k_2||q_2|}{|k_1||q_3|} + \ldots$$

Functions of 4-dimensional anharmonic ratios

$$i\delta_{2\to 4} = \frac{a^2}{4} Li_2(\chi) \ln \frac{\chi t_2 s_{13}}{s_3 t_1} \ln \frac{\chi t_2 s_{02}}{t_3 s_1} + \ldots , \;\; \chi = 1 - \frac{s s_2}{s_{012} s_{123}}$$

12 Multi-gluon states in octet channels

n-gluon Mandelstam cut contribution

$$M_{2\rightarrow 2n}^{(n)} = \int \prod_{r=1}^{n-1} d^2 k_r \, \Phi(k_1,..k_{n-1}) \prod_{t=1}^{n} s^{\omega(k_t)} \, \Phi(k_1,..k_{n-1})$$

Physical region for a non-vanishing result

$$s_1, s_2, ..., s_{n-1}, s_{n+1}, ..., s_{2n} < 0 \, , \; s, s_n > 0$$

Schrödinger equation for gluon composite states

$$H\Psi = E\Psi \, , \; \omega(t) = a\left(-\ln\frac{-t}{\mu^2} + \frac{1}{\epsilon}\right) - \frac{a}{2}E \, , \; a = \frac{g^2 N_c}{8\pi^2}$$

Holomorphic separability

$$H = h + h^* \, , \; h = \ln\frac{p_1 p_n}{q^2} + \sum_{r=1}^{n-1} h_{r,r+1}^t$$

13 Möbius invariance in the p-space

Duality transformation and Möbius invariance

$$p_k = z_{k-1,k}\,,\ \ \rho_{k,k+1} = i\frac{\partial}{\partial z_k} = i\partial_k\,,\ \ z_k \to \frac{az_k+b}{cz_k+d}$$

Hamiltonian in the z-space

$$h = \ln(z_1^2\partial_1) - 2\psi(1) + \ln\partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}\,,\ \ z_0 = 0\,,\ \ z_n = \infty\,,$$

$$h_{1,2} = \ln(z_{12}^2\partial_1) + \ln(z_{12}^2\partial_2) - 2\ln z_{12} - 2\psi(1)$$

Normalization conditions

$$||\Psi||_1^2 = \int \frac{d^2 z_{n-1}}{|z_1|^2}\prod_{r=1}^{n-2}\frac{d^2 z_r}{|z_{r,r+1}|^2}|\Psi|^2\,,\ \ ||\Psi||_2^2 = \int \prod_{r=1}^{n-1} d^2 z_r\,\Psi^*\prod_{t=1}^{n-1}|\partial_t|^2\,\Psi$$

14 Integrable open spin chain

Helpful identity

$$[L_k(u)L_{k+1}(u), h_{k,k+1}] = -i\left(L_k(u) - L_{k+1}(u)\right)$$

Integrals of motion: $[D, h] = 0$

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k} q'_k\,,\ \ q'_k = -\sum_{0<r_1<\ldots<r_k<n} z_{r_1} \prod_{s=1}^{k-1} z_{r_s, r_{s+1}} \prod_{t=1}^{k} i\partial_{r_t}$$

Sklyanin ansatz and Baxter equation

$$\Omega = \prod_{k=1}^{n-2} Q(\hat{u}_k)\,\Omega_0\,,\ \ \Omega_0 = \prod_{l=1}^{n-1} \frac{1}{|z_l|^4}\,,\ \ B(\hat{u}_k) = 0\,,$$

$$D(u)Q(u) = (u+i)^{n-1}Q(u+i)$$

15 Hamiltonian and integrals of motion

Baxter function

$$Q(u) = \prod_{l=1}^{n-2} \frac{\Gamma(-iu - a_l)}{\Gamma(-iu + 1)}\,,\;\; D(u) = \prod_{l=1}^{n-2} (u - ia_l)$$

Constraints for parameters $a_l, \widetilde{a}_l$

$$a_l = i\nu_l + \frac{n_l}{2}\,,\;\; \widetilde{a}_l = i\nu_l - \frac{n_l}{2}$$

Separability of h' at large relative scales

$$h' = \sum_{l=1}^{n-1} \left(\psi(z_l\partial_l) + \psi(-z_l\partial_l) - 2\psi(1)\right)\,,\;\; (z_1 << z_2 << ...z_{n-1})$$

Separability of the holomorphic energy

$$\epsilon = \sum_{l=1}^{n-1} \epsilon(a_r)\,,\;\; \epsilon(a_r) = \psi(a_l) + \psi(-a_l) - 2\psi(1)$$

16 Three-gluon composite state

Wave function in the dual representation

$$\Psi = z_2^{a_1+a_2}(z_2^*)^{\widetilde{a}_1+\widetilde{a}_2} \int \frac{d^2y}{|y|^2}\, y^{-a_2}\, (y^*)^{\widetilde{a}_2} \left(\frac{y-1}{y-z_2/z_1}\right)^{a_1} \left(\frac{y^*-1}{y^*-z_2^*/z_1^*}\right)^{\widetilde{a}_1}$$

Fourier transformation

$$\Psi(\vec{z}_1,\vec{z}_2) = \int d^2p_1\, d^2p_2\, \exp(i\vec{p}_1\vec{z}_1)\, \exp(i\vec{p}_2\vec{z}_2)\, \Psi(\vec{p}_1,\vec{p}_2)\,, \;\; E = E(a_1)+E(a_2)$$

Baxter-Sklyanin representation

$$\Psi^t(\vec{p}_1,\vec{p}_2) = P^{-a_1-a_2}\,(P^*)^{-\widetilde{a}_1-\widetilde{a}_2} \int d^2u\, u\, \widetilde{u}\, Q(u,\widetilde{u}) \left(\frac{p_1}{p_2}\right)^{u} \left(\frac{p_1^*}{p_2^*}\right)^{u^*}$$

Baxter function

$$Q(u,\widetilde{u}) = \frac{\Gamma(-u)\,\Gamma(-\widetilde{u})}{\Gamma(1+u)\,\Gamma(1+\widetilde{u})}\, \frac{\Gamma(u-a_1)\,\Gamma(u-a_2)}{\Gamma(1-\widetilde{u}+\widetilde{a}_1)\,\Gamma(1-\widetilde{u}+\widetilde{a}_2)}\,, \;\; \int d^2u = \int d\nu \sum_n$$

17 Discussion

1. Steinmann relations and analytic properties.
2. Integrability of BFKL dynamics in LLA.
3. BDS amplitudes and Mandelstam cuts.
4. Two loop contribution to $M_{2\to 4}$ from analyticity.
5. Two reggeon state and dual Möbius and conformal invariance.
6. Integrable open spin chain for multi-reggeon composite states.
7. Baxter function and energy separability.